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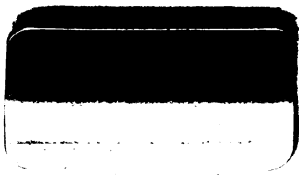
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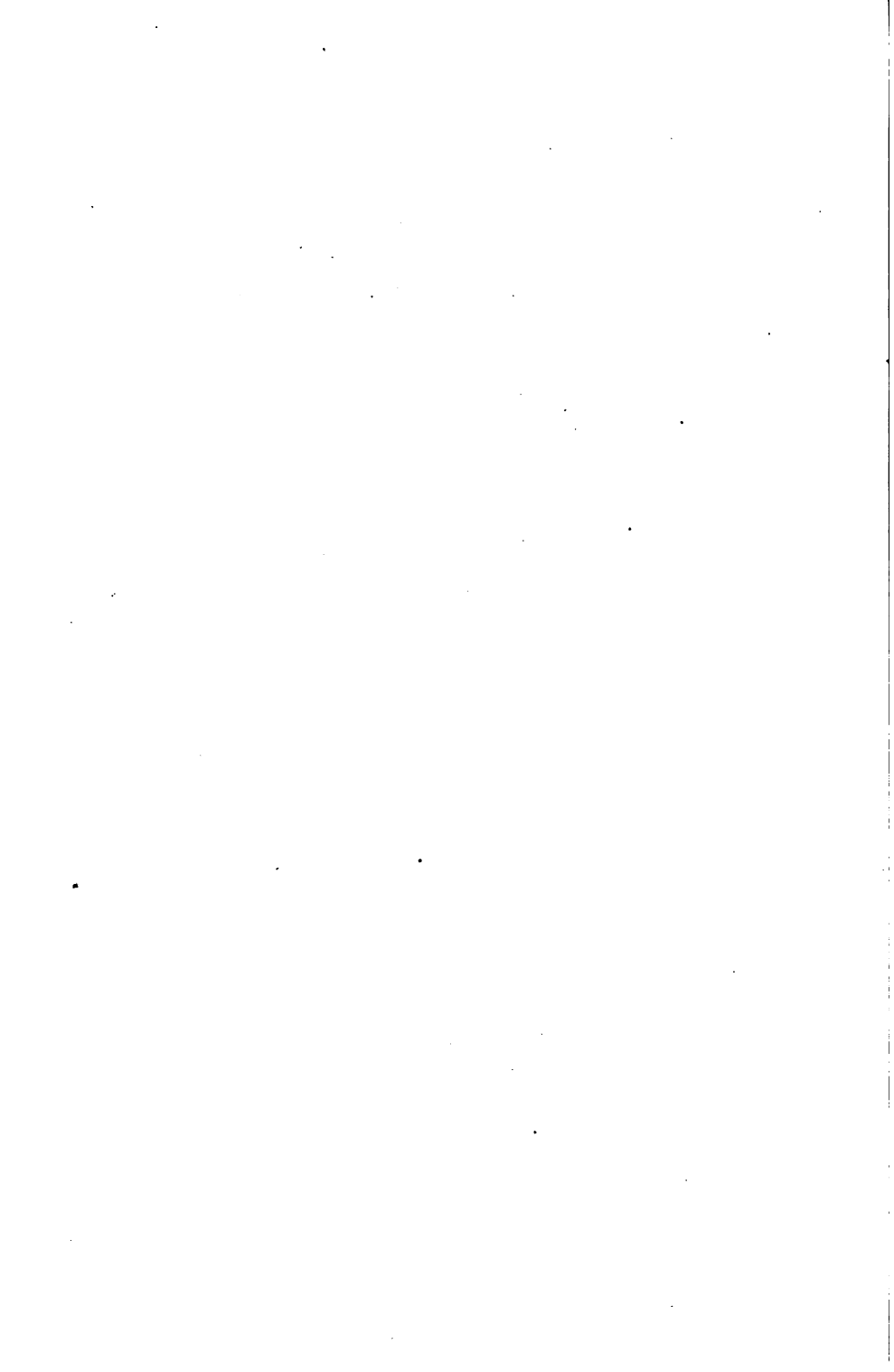
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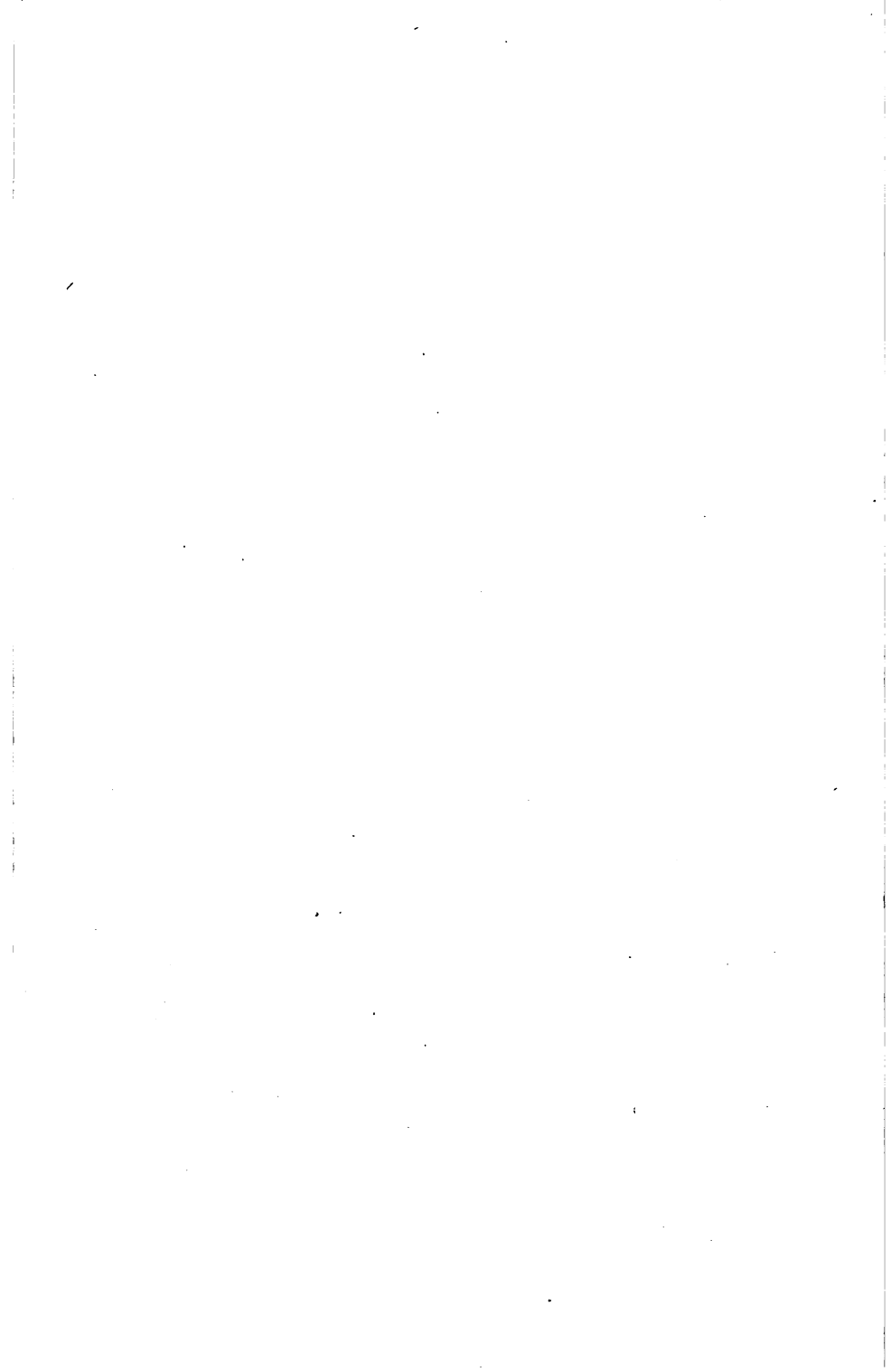
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Rev. W.



Lips  
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# ELEMENTS OF GEOMETRY

BY

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AND

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PROFESSORS IN YALE UNIVERSITY

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PART TWO—GEOMETRY OF SPACE

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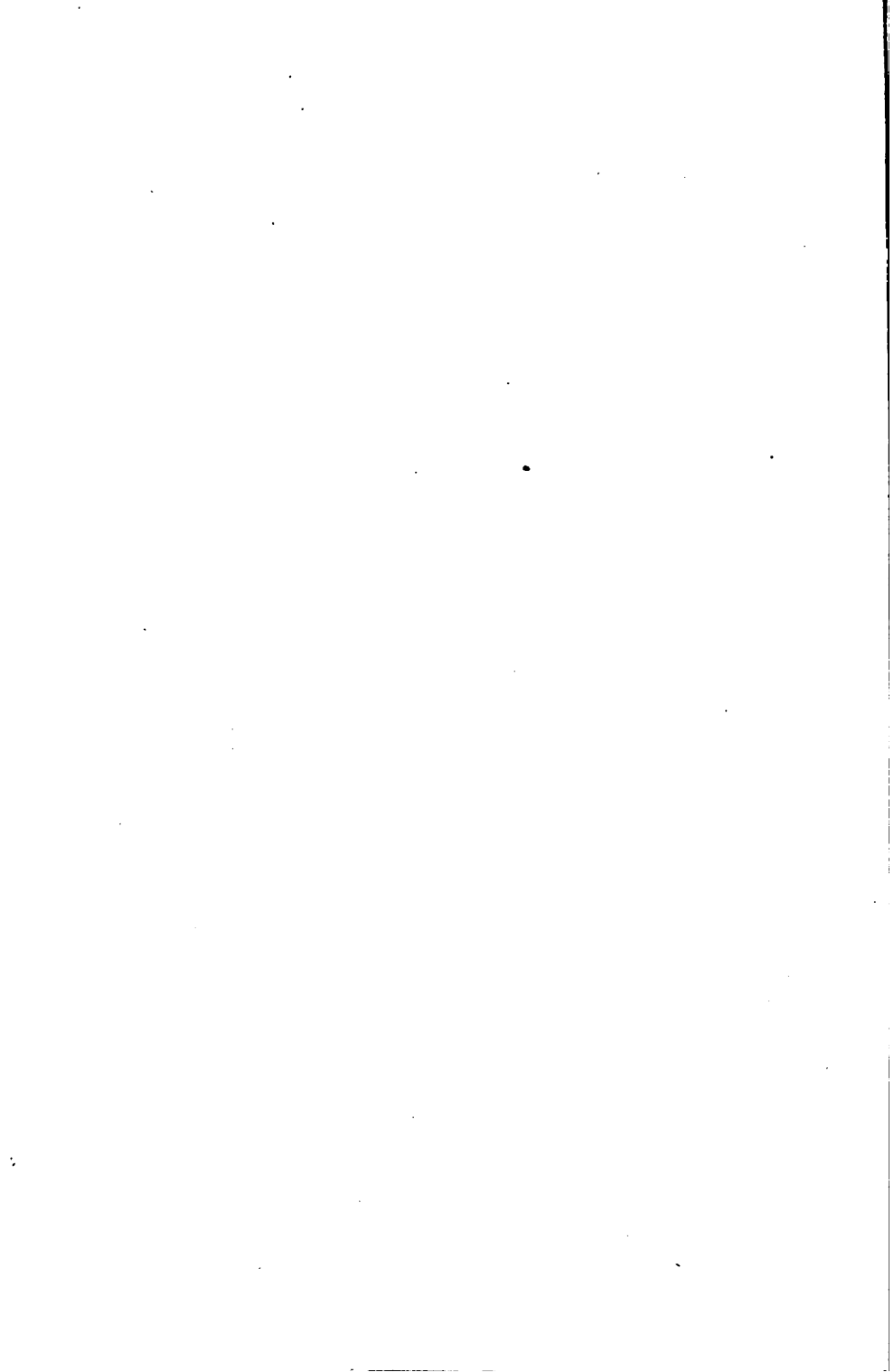
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## SPECIAL TERMS

An **axiom** is a truth assumed as self-evident.

A **theorem** is a truth which becomes evident by a train of reasoning called a **demonstration**.

A theorem consists of two parts, the *hypothesis*, that which is given, and the *conclusion*, that which is to be proved.

A **problem** is a question proposed which requires a solution.

A **proposition** is a general term for either a theorem or problem.

One theorem is the **converse** of another when the conclusion of the first is made the hypothesis of the second, and the hypothesis of the first is made the conclusion of the second.

The converse of a truth is not always true. Thus, "If a man is in New York City he is in New York State," is true; but the converse, "If a man is in New York State he is in New York City," is not necessarily true.

When one theorem is easily deduced from another the first is sometimes called a **corollary** of the second.

A theorem used merely to prepare the way for another theorem is sometimes called a **lemma**.

## SYMBOLS AND ABBREVIATIONS

+ plus.	Cons.—Construction.
— minus.	Cor.—Corollary.
> is greater than.	Def.—Definition.
< is less than.	Fig.—Figure.
× multiplied by.	Hyp.—Hypothesis.
= equals.	Iden.—Identical.
⊆ is equivalent to.	Q. E. D.—Quod erat demonstrandum.
Alt.-int.—Alternate interior.	Q. E. F.—Quod erat faciendum.
Ax.—Axiom.	Sup.-adj.—Supplementary adjacent.





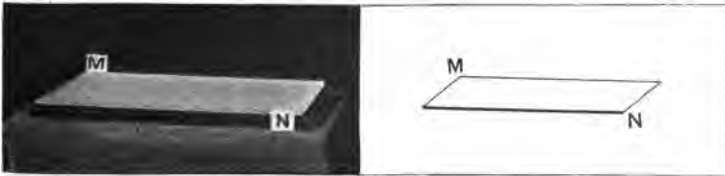
# GEOMETRY OF SPACE

## BOOK VI

### STRAIGHT LINES AND PLANES

**524. Def.**—A plane has already been defined as “a surface such that, if any two points in it are taken, the straight line passing through them lies wholly in the surface.” § 8

A plane is regarded as indefinite in extent, but is usually represented to the eye by a parallelogram lying in it.



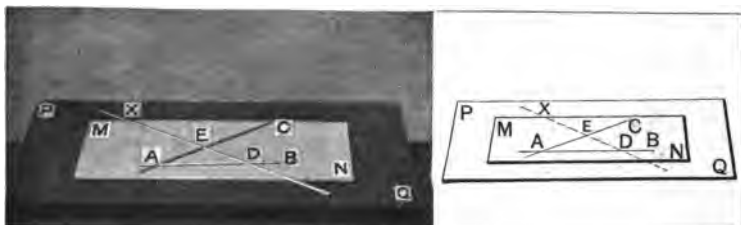
**525. Def.**—A plane is **determined** by given conditions, if it is the only plane fulfilling these conditions.



## PROPOSITION I. THEOREM

**526.** *A plane is determined if it passes through:*

- I. *Three points not in the same straight line.*
- II. *A straight line and a point without that line.*
- III. *Two intersecting straight lines.*
- IV. *Two parallel straight lines.*



I. GIVEN—three points,  $A$ ,  $B$ , and  $C$  not in the same straight line.

TO PROVE—that one and only one plane can be passed through them.

Pass a plane  $MN$  through one of the points, turn it about this point until it contains one of the other points, and then turn it about these two points until it contains the third.

No other plane will contain these points.

For, suppose  $PQ$  to be such a plane.

Take  $X$  any point in  $PQ$ . We will prove it also lies in  $MN$ .

Draw the straight lines  $AB$  and  $AC$ .

These will be in both planes, since  $A$ ,  $B$ , and  $C$  lie in both planes. § 524

Through  $X$  draw a straight line in  $PQ$  cutting  $AB$  and  $AC$  in  $D$  and  $E$ .

Since  $D$  and  $E$  lie in the plane  $MN$ , the straight line  $DEX$  lies wholly in  $MN$ . § 524

Hence  $X$ , a point in  $DE$ , lies in the plane  $MN$ .

Thus *any* point, that is, *every* point in the plane  $PQ$  lies also in the plane  $MN$ , and in like manner we can prove that every point in  $MN$  lies in  $PQ$ .

Therefore the two planes coincide.

Q. E. D.

II. GIVEN—the straight line  $AB$  and the point  $C$  without  $AB$ .

TO PROVE—that one and only one plane can be passed through them,

The plane passed through  $C$  and any two points of  $AB$  will contain  $AB$ . § 524

We can pass no other plane through  $AB$  and  $C$ , for then we should have two planes containing three points not in the same straight line, which is impossible. Q. E. D.

III. GIVEN—the straight lines  $AB$  and  $AC$  intersecting in  $A$ .

TO PROVE—that one and only one plane can be passed through them.

The plane passed through the three points  $A$ ,  $B$ , and  $C$  will contain the straight lines  $AB$  and  $AC$ . § 524

We can pass no other plane through  $AB$  and  $AC$ , for then we should have two planes containing three points not in the same straight line, which is impossible. Q. E. D.



IV GIVEN—the parallel straight lines  $FG$  and  $KL$ .

TO PROVE—that one and only one plane can be passed through them.

By definition these parallel lines lie in the same plane.

§ 31

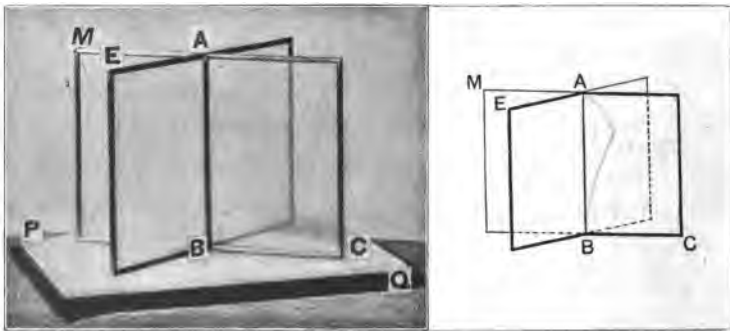
There cannot be two planes passed through them, for then we would have two planes containing three points  $F$ ,  $G$ , and  $K$ , not in the same straight line, which is impossible.

Q. E. D.

**527. Def.**—The intersection of two planes is the line common to both planes.

PROPOSITION II. THEOREM

**528.** *If two planes intersect, their intersection is a straight line.*



GIVEN two intersecting planes,  $MB$  and  $EB$ .

TO PROVE their intersection is a straight line.  $\therefore$

If possible, suppose the intersection is not straight.

It would then contain three points not in the same straight line.

That is, the two planes would contain three points not in the same straight line, which is impossible. § 526 I

Therefore the intersection must be a straight line. Q. E. D.

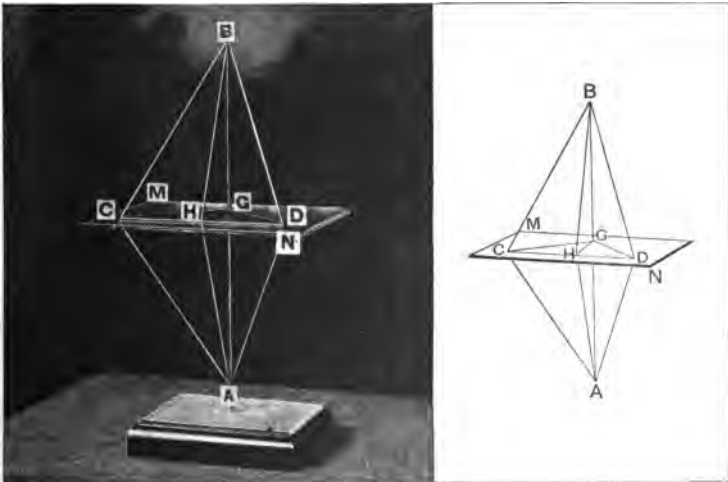
PERPENDICULAR AND OBLIQUE LINES AND PLANES

**529. Def.**—If a straight line meet a plane, its point of meeting is called its foot.

**530. Defs.**—A straight line is **perpendicular** to a plane, if it is perpendicular to every straight line in the plane drawn through its foot. In the same case the plane is said to be perpendicular to the line.

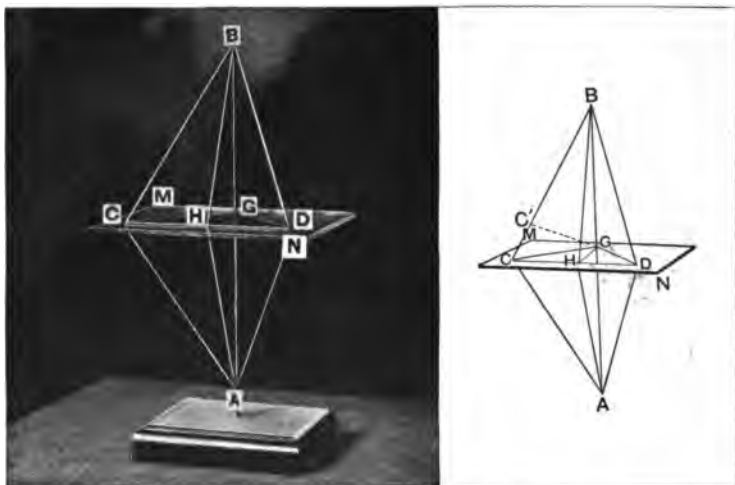
PROPOSITION III. THEOREM

**531.** *If two intersecting straight lines are perpendicular to a third at the same point, their plane is perpendicular to that straight line.*



**GIVEN**—the two intersecting straight lines  $GC$  and  $GD$  perpendicular to the straight line  $BG$  at the point  $G$ .

**TO PROVE**—that the plane  $MN$  passed through  $GC$  and  $GD$  is perpendicular to  $BG$ .



In the plane  $MN$  draw through  $G$  any straight line  $GH$ .

Let  $CD$  be any straight line cutting the lines  $GC$ ,  $GH$ , and  $GD$  in  $C$ ,  $H$ , and  $D$ .

Produce the line  $BG$  to  $A$  making  $GA$  equal to  $GB$ , and join  $A$  and  $B$  to  $C$ ,  $H$ , and  $D$ .

Then, since  $GC$  is perpendicular to  $BA$  at its middle point,

$$CB = CA. \quad \S 103$$

Similarly  $DB = DA.$

Hence the triangles  $BCD$  and  $ACD$  are equal.  $\S 89$

And the homologous angles  $BCH$  and  $ACH$  are equal.

Hence the triangles  $BCH$  and  $ACH$  are equal.  $\S 79$

Therefore their homologous sides  $BH$  and  $AH$  are equal.

Therefore  $GH$  is perpendicular to  $BA.$   $\S 104$

But  $GH$  is any straight line in  $MN$  passing through  $G$ .

Therefore *every* straight line in  $MN$  passing through  $G$  is perpendicular to  $BA$ ; that is,  $MN$  is perpendicular to  $BA$ .

§ 530  
Q. E. D.

**532. COR. I.** *At a given point in a straight line one and only one plane can be drawn perpendicular to that straight line.*

*Hint.*—Let  $AB$  be the straight line and  $G$  the point.

At  $G$  draw the straight lines  $GC$  and  $GD$  perpendicular to  $AB$ .

The plane of these lines will be perpendicular to  $AB$ . (Why?)

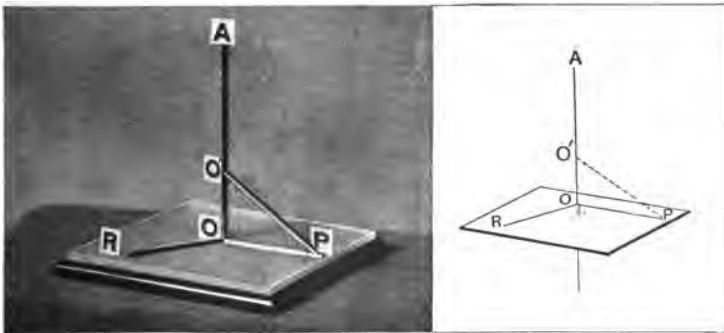
Only one such plane can be drawn.

For any other plane passing through  $G$  cannot contain both of the lines  $GC$  and  $GD$ . (Why?)

It must therefore cut one of the planes  $BGC$  and  $BGD$ , say  $BGC$ , in some line  $GC'$  other than  $GC$  and  $GD$ .

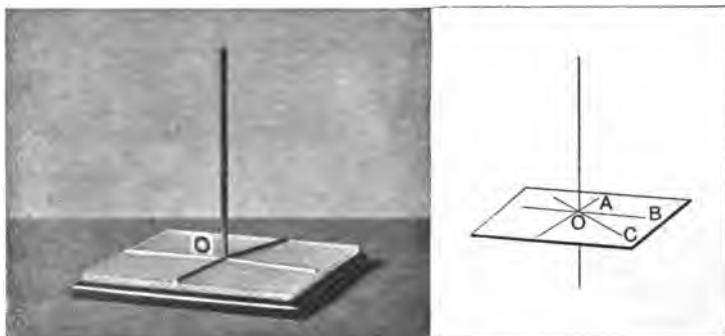
Since  $BGC'$  is not a right angle, this second plane is not perpendicular to  $AB$ . (Why?)

**533. COR. II.** *Through a given point without a straight line one and only one plane can be passed perpendicular to that straight line.*



*Hint.*—Use § 531 to draw *one* such plane. Any other plane cuts  $AO$  either at  $O$  or at some other point,  $O'$ . § 532 shows that the first is not perpendicular. Show also that the second is not.

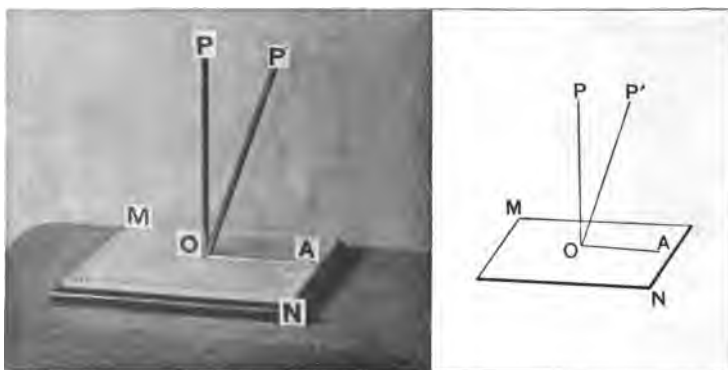
**534.** COR. III. *All the perpendiculars to a given straight line at the same point lie in a plane perpendicular to that line at that point.*



*Hint.*—Every pair of these perpendiculars, as  $OA$  and  $OB$ , determines a plane perpendicular at  $O$ . (Why?)

And all the planes thus determined must coincide. (Why?) Hence, etc.

**535.** COR. IV. *At a point in a plane one and only one perpendicular to the plane can be drawn.*



*Hint.*—Prove from Corollary I. that one perpendicular  $OP$  to the plane  $MN$  can be drawn.

No other line, as  $OP'$  through  $O$  can be perpendicular to  $MN$ .

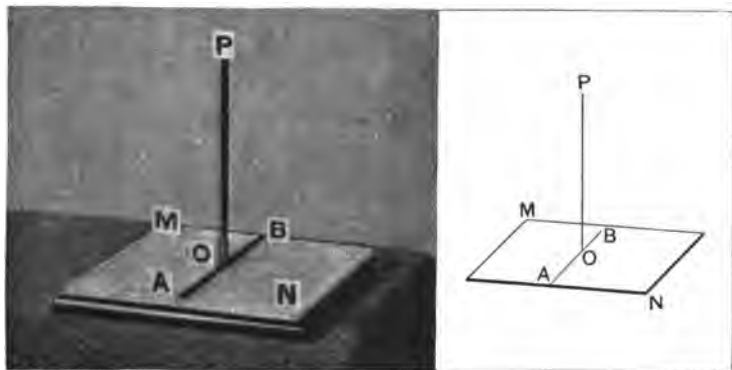
For, let the plane of  $OP$  and  $OP'$  intersect  $MN$  in  $OA$ .

Since  $OP$  is perpendicular to  $OA$ ,  $OP'$  is not. (Why?)

Therefore  $OP'$  is not perpendicular to  $MN$ . (Why?)

PROPOSITION IV. THEOREM

**536.** *The minimum line from a point to a plane is perpendicular to that plane.*



**GIVEN**—the plane  $MN$  and the point  $P$  without it, and  $PO$ , the minimum line from  $P$  to  $MN$ .

**TO PROVE**—that  $PO$  is perpendicular to  $MN$ .

In the plane  $MN$  through the point  $O$  draw *any* straight line  $AB$ .

Since  $PO$  is the shortest line from  $P$  to the plane  $MN$ , it is the shortest line from  $P$  to the line  $AB$  in that plane.

Therefore  $PO$  is perpendicular to  $AB$ . § 96

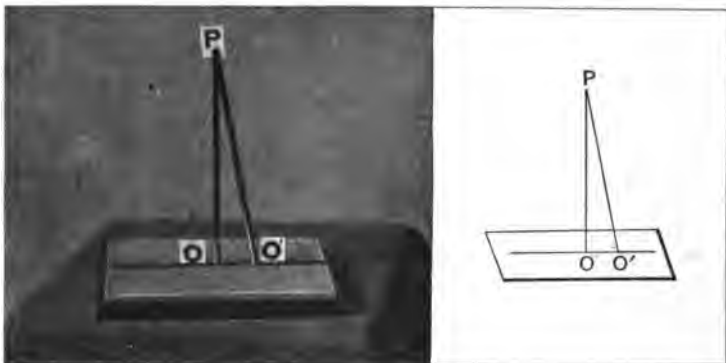
That is,  $PO$  is perpendicular to *any* or *every* straight line in  $MN$  through its foot  $O$ .

Therefore  $PO$  is perpendicular to  $MN$ .

Q. E. D.



**537. COR.** *From a point without a plane one and only one perpendicular to the plane can be drawn.*



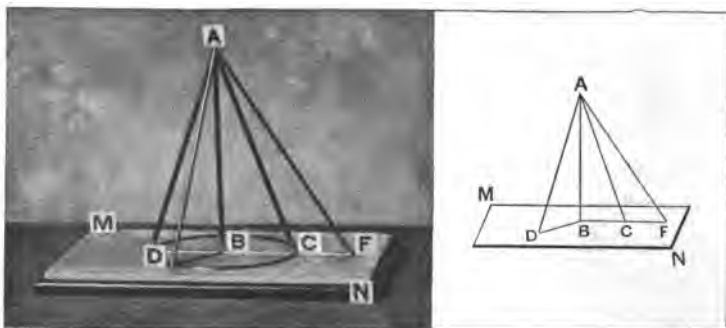
*Hint.*—Apply the Proposition and § 34.

**538. Def.**—By the distance from a point to a plane is meant the shortest distance, and therefore the perpendicular distance.

PROPOSITION V. THEOREM

**539.** *If oblique lines are drawn from a point to a plane :*

- I. *Those meeting the plane at equal distances from the foot of the perpendicular are equal.*
- II. *Of two unequally distant, the more remote is the greater.*



I. GIVEN—the oblique lines  $AC$  and  $AD$  meeting the plane  $MN$  at the equal distances  $BC$  and  $BD$  from the perpendicular  $AB$ .

TO PROVE  $AC=AD$ .

In the triangles  $ABC$  and  $ABD$ ,  $AB$  is common;  $BC=BD$  by hypothesis; and the angles  $ABC$  and  $ABD$  are equal, being right angles.

Therefore the triangles are equal, and  $AC=AD$ : § 79  
Q. E. D.

II. GIVEN—the oblique lines  $AF$  and  $AD$  meeting  $MN$  so that

$$BF > BD$$

TO PROVE  $AF > AD$

On  $BF$  take  $BC=BD$  and draw  $AC$ .

Then, from *plane* geometry,  $AF > AC$ . § 99

But  $AD=AC$ . Case I

Therefore  $AF > AD$ . Q. E. D.

**540.** COR. Conversely:

I. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular.*

II. *Of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

*Hint.*—Prove as in § 100.

**541.** Remark.—Article 540 supplies practical methods of drawing a straight line perpendicular to a plane, as a floor or a blackboard.

I. *To erect a perpendicular to a plane at a given point in it.*

With the given point as centre, describe a circumference in the given plane.

Take three strings of equal length somewhat longer than the radius of the circumference.

To each of three points on the circumference attach an end of one string.

Unite the three remaining ends in a knot and pull the strings taut.

A line through the given point and the knot is the perpendicular required. Prove the method correct by supposing if possible that the foot of the perpendicular from the knot is not in the given point, and apply § 103.

II. *To draw a perpendicular to a given plane from a given point without it.*

From the point with a string of convenient length measure three equal distances to the plane.

The centre of the circumference which passes through the three points thus found is the foot of the required perpendicular. (Why?)

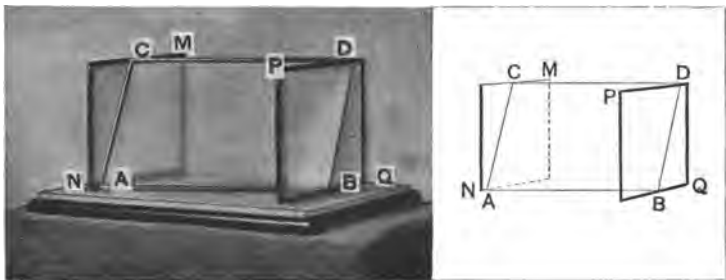
#### PARALLEL LINES AND PLANES

**542. Def.**—A straight line and a plane are **parallel** to each other if they cannot meet, however far produced.

**543. Def.**—Two planes are **parallel** to each other if they cannot meet, however far produced.

#### PROPOSITION VI. THEOREM

**544.** *If two parallel planes are cut by a third plane, their intersections with this plane are parallel.*



GIVEN—the parallel planes  $MN$  and  $PQ$  cut by the plane  $AD$  in the lines  $AC$  and  $BD$ .

TO PROVE  $AC$  and  $BD$  parallel.

Since the planes  $MN$  and  $PQ$  cannot meet, the lines  $AC$  and  $BD$  lying in them cannot meet.

Moreover these lines lie in the same plane  $AD$ .

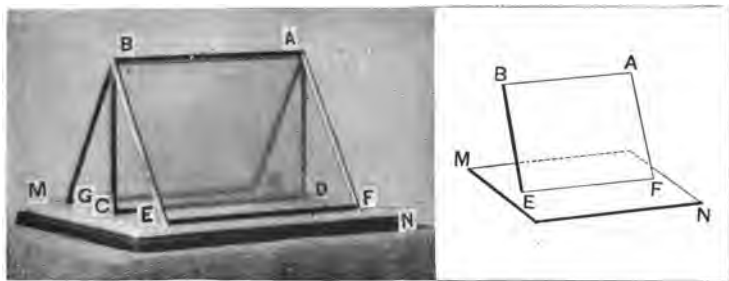
Therefore they are parallel.

§ 31  
Q. E. D.

**545.** COR. *Parallel lines  $AB$  and  $CD$  intercepted between parallel planes are equal.*

PROPOSITION VII. THEOREM

**546.** *If a straight line is parallel to a plane, the intersection of the plane with a plane passed through the line is parallel to the line.*



GIVEN—the line  $BA$  parallel to the plane  $MN$  and a plane  $BF$  passing through  $BA$  and intersecting  $MN$  in  $EF$ .

TO PROVE  $BA$  parallel to  $EF$ .

These lines lie in the same plane.

They cannot meet, for  $BA$  cannot meet the plane  $MN$  in which  $EF$  lies.

Therefore they are parallel.

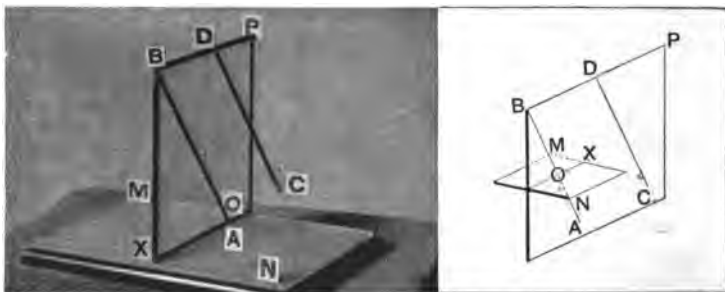
§ 31  
Q. E. D.

**547. COR.** *If two intersecting straight lines are parallel to a plane, their plane is parallel to the given plane.*

*Hint.*—If their plane were not parallel to the given plane it would intersect it in a line which would be parallel to both the given lines.

PROPOSITION VIII. THEOREM

**548.** *A plane which cuts one of two parallel lines must, if sufficiently produced, cut the other also.*



**GIVEN**—the parallel lines  $AB$  and  $CD$ , one of which,  $AB$ , is cut by the plane  $MN$  in the point  $O$ .

**TO PROVE** that  $CD$  is also cut by  $MN$ .

Pass a plane through  $AB$  and  $CD$ .

As this plane and the plane  $MN$  have the point  $O$  in common, their intersection must contain  $O$ . Call it  $OX$ .

Now suppose, if possible, that  $MN$  does not cut the line  $CD$ , but is parallel to it.

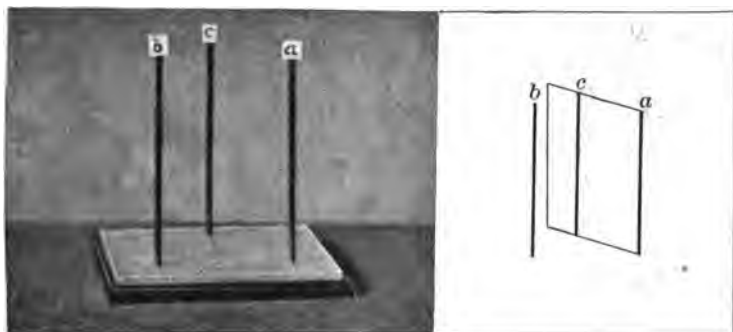
Then  $OX$  will also be parallel to  $CD$ . § 546

And there will be *two* lines,  $OX$  and  $OB$  through  $O$ , parallel to  $CD$ , which is impossible.

Therefore  $MN$  must cut  $CD$ .

Q. E. D.

**549. COR. I.** *If two straight lines  $a$  and  $c$  are parallel to a third  $b$ , they are parallel to each other.*

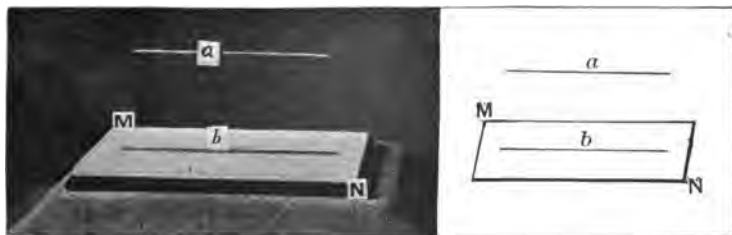


*Hint.*—Pass a plane through  $a$  and any point of  $c$ .

This plane will entirely contain  $c$ . Otherwise it would cut  $c$  and therefore  $b$ , which is parallel to  $c$ , and also  $a$ , which is parallel to  $b$ . This contradicts the hypothesis that it contains  $a$ .

Prove also that  $a$  and  $c$  cannot meet.

**550.** COR. II. *If two straight lines  $a$  and  $b$  are parallel, any plane  $MN$ , that contains one, as  $b$ , and not the other, is parallel to the second.*



*Hint.*—If  $MN$  is not parallel to  $a$ , it will cut it.

This is impossible, for then  $MN$  would cut  $b$  also.

Therefore  $MN$  is parallel to  $a$ .

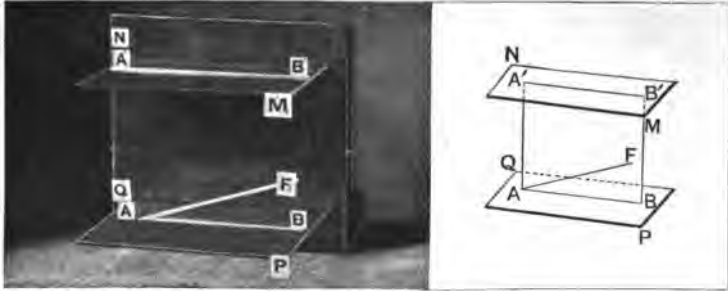
**551.** COR. III. *If two intersecting straight lines are parallel to two other intersecting straight lines, the plane of the first pair is parallel to the plane of the second pair.*

*Hint.*—Apply § 550 and then § 547.

## PROPOSITION IX. THEOREM

**552.** *If two planes are parallel:*

- I. *Any straight line that cuts one cuts the other.*
- II. *Any plane that cuts one cuts the other.*



- I. **GIVEN**—the parallel planes  $MN$  and  $PQ$  and the straight line  $AF$  cutting  $PQ$  in the point  $A$ .

**TO PROVE**—that  $AF$  is not parallel to  $MN$  but cuts  $MN$ .

Through  $AF$  and any point  $A'$  of  $MN$  not in  $AF$  pass a plane  $A'B$ .

Since this plane has a point in common with each of the parallel planes, it will intersect each in straight lines  $AB$  and  $A'B'$ .

These lines will be parallel. § 544

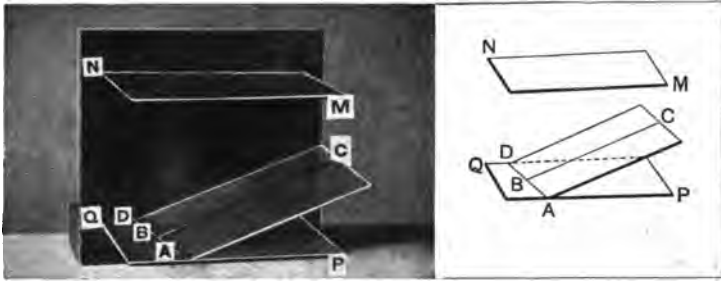
In the plane  $A'B$  we have  $AF$  cutting  $AB$ , one of the two parallels  $AB$  and  $A'B'$ .

It therefore cuts the other,  $A'B'$ , since  $AF$  and  $AB$  cannot both be parallel to  $A'B'$ . Ax.  $b$

Therefore  $AF$  cutting  $A'B'$  cuts the plane  $MN$  in which  $A'B'$  lies. Q. E. D.

- II. **GIVEN**—the plane  $CD$  intersecting  $PQ$  in the straight line  $AD$ .

**TO PROVE** that  $CD$  also intersects  $MN$ .



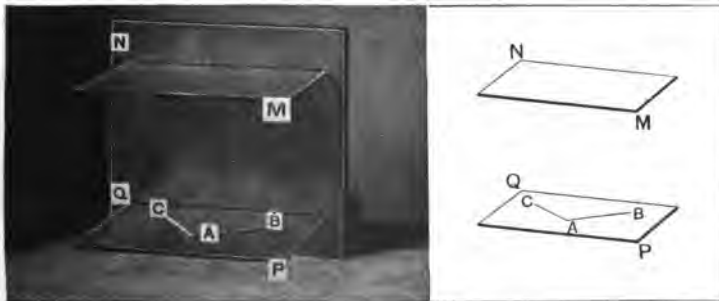
In the plane  $CD$  draw any straight line  $BC$  cutting  $AD$ .  
 This line cuts  $PQ$ , and therefore cuts  $MN$ , by the first part of the proposition.

Therefore the plane  $CD$ , in which  $BC$  lies, will cut  $MN$ .

Q. E. D.

**553.** COR. I. *If two planes are parallel to a third plane they are parallel to each other.*

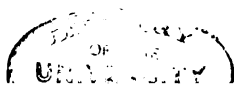
**554.** COR. II. *Through a given point without a given plane there can be drawn a plane parallel to the given plane, and but one.*



*Hint.*—Through the point  $A$ , without the plane  $MN$ , draw two straight lines  $AB$  and  $AC$  parallel to  $MN$ .

$PQ$ , the plane of  $AB$  and  $AC$ , will be parallel to  $MN$ .

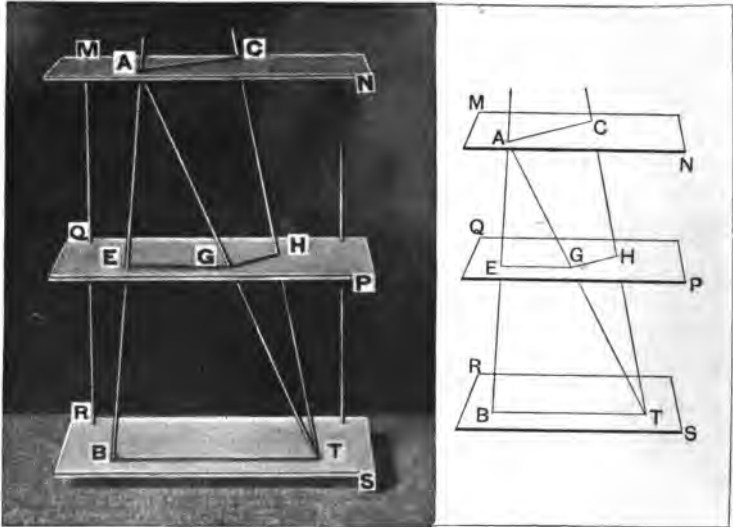
No other plane through  $A$  could be parallel to  $MN$ , for it would cut  $PQ$  and therefore also  $MN$ .





## PROPOSITION X. THEOREM

**555.** *If two straight lines are cut by three parallel planes, their corresponding segments are proportional.*



GIVEN—the straight lines  $AB$  and  $CT$  cut by the parallel planes  $MN$ ,  $PQ$ , and  $RS$  in the points  $A, E, B$ , and  $C, H, T$ .

TO PROVE 
$$\frac{AE}{EB} = \frac{CH}{HT}.$$

Join  $A$  to  $T$  by a straight line cutting  $PQ$  in  $G$ .

Draw  $EG, BT, GH$ , and  $AC$ .

Then  $EG$  and  $GH$  will be parallel to  $BT$  and  $AC$  respectively. § 544

Therefore 
$$\frac{AE}{EB} = \frac{AG}{GT}, \text{ and } \frac{AG}{GT} = \frac{CH}{HT}. \quad \text{§ 271}$$

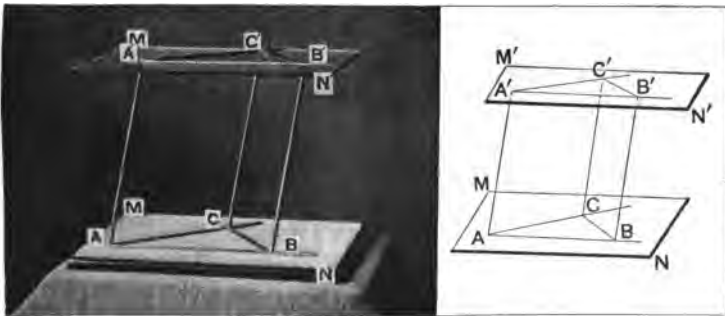
Hence 
$$\frac{AE}{EB} = \frac{CH}{HT}. \quad \text{Ax. I}$$
Q. E. D.

**556. COR.** *If a series of lines passing through a common point are cut by two parallel planes, their corresponding segments are proportional.*

*Hint.*—Pass a third plane through the common point parallel to one (and hence the other) of the two given planes.

PROPOSITION XI. THEOREM

**557.** *If two angles not in the same plane have their sides respectively parallel and extending from their vertices in the same direction, they are equal.*



GIVEN—the angles  $BAC$  and  $B'A'C'$ , whose sides,  $AB$ ,  $A'B'$ , and  $AC$ ,  $A'C'$ , are respectively parallel and extending in the same direction.

TO PROVE  $\text{angle } BAC = \text{angle } B'A'C'$ .

Take  $AB = A'B'$  and  $AC = A'C'$  and join  $AA'$ ,  $BB'$ ,  $CC'$ .

Then  $AB'$  and  $AC'$  will be parallelograms. § 126

Hence  $BB'$  and  $CC'$  are equal to and parallel to  $AA'$ .

§§ 117, 114

Hence  $BB'$  and  $CC'$  are equal to and parallel to each other. Ax. 1, § 549

Therefore  $BC'$  is a parallelogram, and  $BC = B'C'$ . § 126

The triangles  $ABC$  and  $A'B'C'$  are therefore equal. § 89

Hence  $\text{angle } BAC = \text{angle } B'A'C'$ . Q. E. D.

**558.** COR. *If two angles not in the same plane have their sides respectively parallel and extending in opposite directions from their vertices, they are equal; if two corresponding sides extend in the same direction, and the other two in opposite directions, the angles are supplementary.*

PROPOSITION XII. THEOREM

**559.** *If two planes are perpendicular to the same straight line, they are parallel.*



GIVEN—the planes  $b$  and  $c$  perpendicular to the straight line  $a$ .

TO PROVE  $b$  and  $c$  parallel.

If they should meet, we should have through any point of their intersection two planes,  $b$  and  $c$ , perpendicular to the same straight line  $a$ .

This is impossible.

§ 533

Therefore  $b$  and  $c$  are parallel.

Q. E. D.

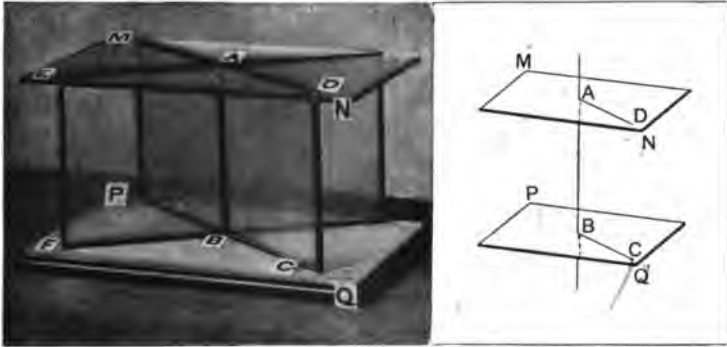
**560.** *Exercise.*—Prove this proposition as a consequence of §§ 33, 51.

*Hint.*—Pass two planes through  $a$  intersecting  $b$  and  $c$  in straight lines perpendicular to  $a$ .

✓

PROPOSITION XIII. THEOREM

**561.** *If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other.*



**GIVEN**—the parallel planes  $MN$  and  $PQ$ , and the line  $AB$  perpendicular to  $MN$  at  $A$ .

**TO PROVE**  $AB$  perpendicular to  $PQ$ .

Since  $AB$  cuts  $MN$ , it also cuts  $PQ$  in some point  $B$ . § 552 I  
 [If two planes are parallel, any line that cuts one cuts the other.]

Through  $B$  draw in  $PQ$  any straight line  $BC$ , and through  $AB$  and  $BC$  pass a plane intersecting  $MN$  in  $AD$ .

Then  $BC$  is parallel to  $AD$ , § 544  
 [If two planes are parallel, their intersections with a third plane are parallel.]

But  $AB$  is perpendicular to  $AD$ . § 530  
 [A straight line perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.]

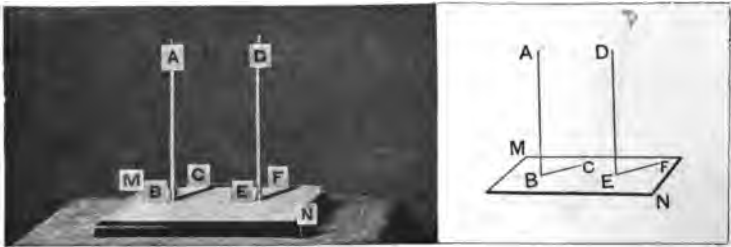
Therefore  $AB$  is also perpendicular to  $BC$ . § 36

Since  $AB$  is perpendicular to *any* straight line drawn in  $PQ$  through  $B$ , it is perpendicular to  $PQ$ . § 530

Q. E. D

## PROPOSITION XIV. THEOREM

**562.** *If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other.*



**GIVEN**—the parallel lines  $AB$  and  $DE$  and the plane  $MN$  perpendicular to  $AB$  at  $B$ .

**TO PROVE**  $MN$  perpendicular to  $DE$ .

Since  $MN$  cuts  $AB$ , it also cuts  $DE$  in some point  $E$ . § 548  
 [If two lines are parallel, any plane that cuts one cuts the other.]

Through  $E$  draw in  $MN$  any straight line  $EF$ , and through  $B$  draw in  $MN$  the line  $BC$  parallel to  $EF$ .

Then angle  $DEF = \text{angle } ABC$ . § 557

But, since  $BC$  lies in  $MN$ ,  $ABC$  is a right angle. § 530

Hence  $DEF$  is a right angle.

Since *any* straight line in  $MN$  through  $E$  is perpendicular to  $DE$ ,  $MN$  is perpendicular to  $DE$ . Q. E. D.

**563. COR. I.** *If two straight lines are perpendicular to the same plane, they are parallel.*

*Hint.*—Suppose  $AB$  and  $DE$  perpendicular to  $MN$ .

Through any point of  $DE$  draw a line, as  $ED'$ , parallel to  $AB$ .

Prove that  $DE$  and  $ED'$  coincide.

**564. Exercise.**—Prove § 549 by means of §§ 562, 563.

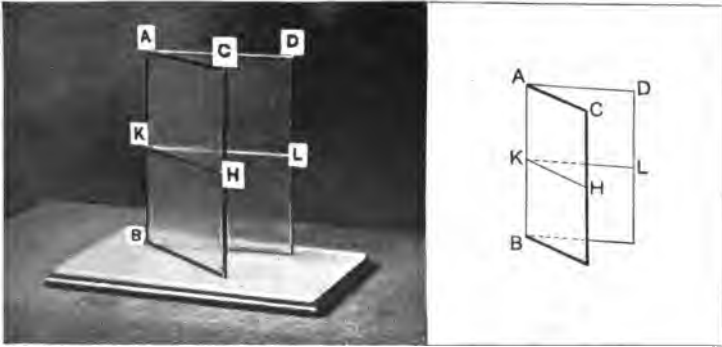
**565. COR. II.** *The perpendicular distance between two parallel planes is everywhere the same.*

DIEDRAL ANGLES AND PROJECTIONS

**566. Defs.**—When two planes meet and are terminated at their common intersection, they are said to form a **diedral angle**.

The planes are called the **faces** of the diedral angle, and their intersection, the **edge**.

The faces are regarded as indefinite in extent.



We may designate a diedral angle by two points on its edge and one other point in each face, the former two being written between the latter two.

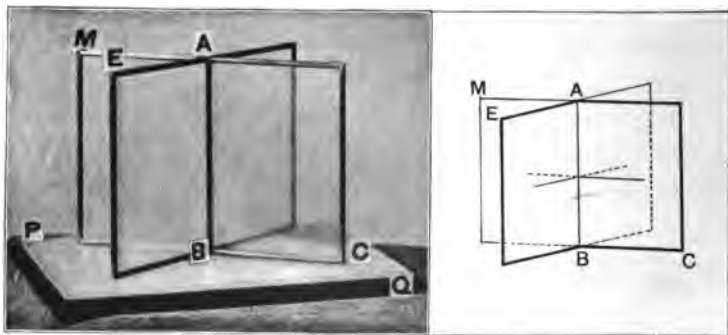
Thus, in the figure, the two planes  $BC$  and  $BD$  meeting in the line  $AB$  form the diedral angle  $CABD$ ;  $BC$  and  $BD$  are the faces of the diedral angle, and  $AB$  is its edge.

If there is only one diedral angle at an edge, it may be designated by two points on its edge; thus the diedral angle  $CABD$  may also be called the diedral angle  $AB$ .

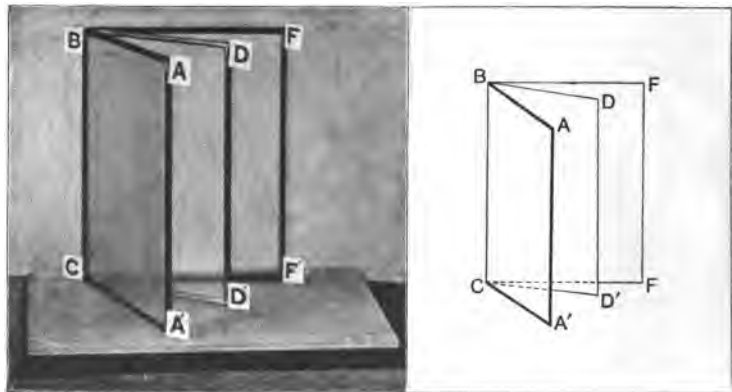
**567. Def.**—The **plane angle** of a diedral angle is the angle formed by two straight lines drawn one in each face of the diedral angle perpendicular to its edge at the same point.

Thus  $HKL$  is the plane angle of the diedral angle  $CABD$ .

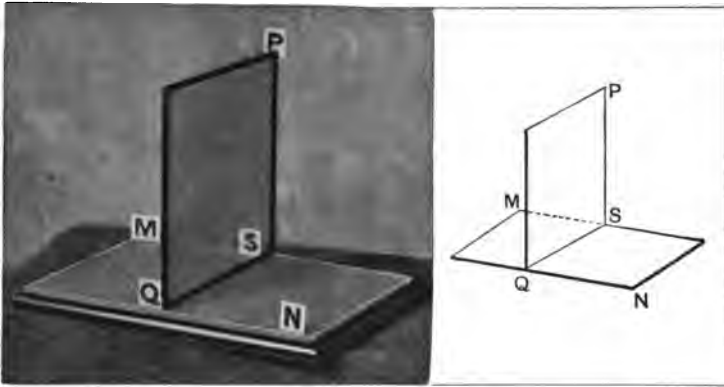
**568. Def.**—Two diedral angles are **vertical** if the faces of one are the prolongations of the faces of the other.



**569. Def.**—Two diedral angles are **adjacent** when they have a common edge and a common face lying between them; as  $ABCD$  and  $FBCD$ .



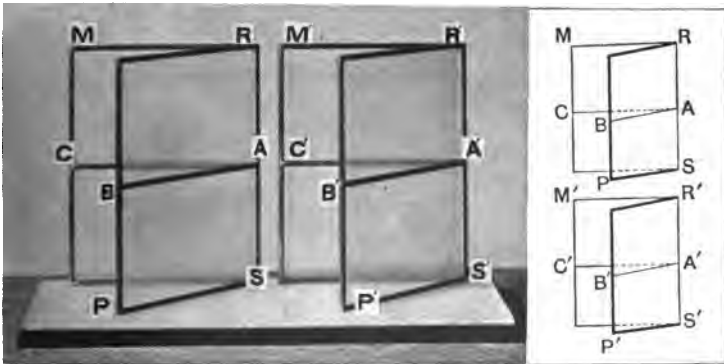
**570. Def.**—If a plane meets another plane so as to form with it two equal adjacent diedral angles, each of these diedral angles is called a **right diedral angle**, and the first plane is said to be **perpendicular** to the second.



Thus the plane  $PQ$  is perpendicular to the plane  $MN$ , if the dihedral angles  $PQSN$  and  $PQSM$  are equal.

PROPOSITION XV. THEOREM

**571.** *If two dihedral angles are equal, their plane angles are equal.*



**GIVEN** the equal dihedral angles  $MRSP$  and  $M'R'S'P'$ .

**TO PROVE** their plane angles  $CAB$  and  $C'A'B'$  equal.



Superpose the dihedral angle  $M'R'S'P'$  upon its equal  $MRSP$ , letting  $A'$  fall at  $A$ .

Then, since  $A'B'$  and  $AB$  are both perpendicular to the line  $RS$  at  $A$  in the plane  $RP$ , they coincide. § 18

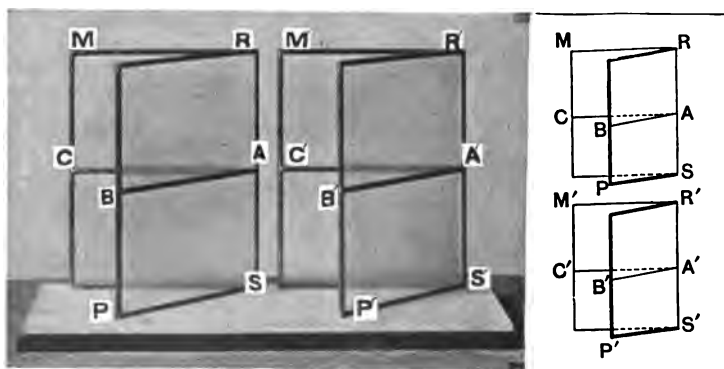
Similarly  $A'C'$  and  $AC$  coincide.

Therefore the angles  $CAB$  and  $C'A'B'$  are equal. § 15  
Q. E. D.

PROPOSITION XVI. THEOREM

**572.** *If the plane angles of two dihedral angles are equal, the dihedral angles are equal.*

[Converse of Proposition XV.]



GIVEN—two dihedral angles,  $MRSP$  and  $M'R'S'P'$ , whose plane angles,  $CAB$  and  $C'A'B'$ , are equal.

TO PROVE the dihedral angles equal.

Since  $RS$  is perpendicular to the lines  $AB$  and  $AC$ , it is perpendicular to their plane. § 531

Similarly  $R'S'$  is perpendicular to the plane of  $A'B'$  and  $A'C'$ .

Place the angle  $C'A'B'$  upon its equal  $CAB$ .

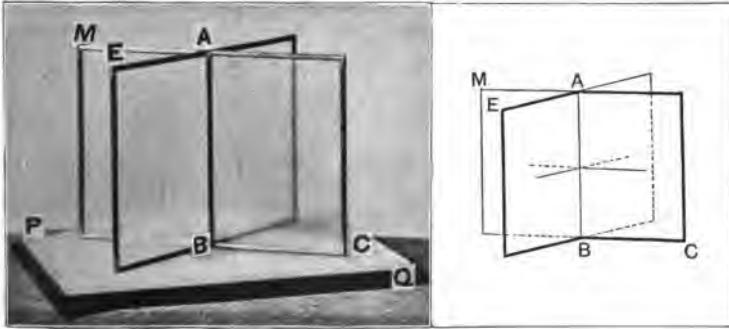
Then  $R'S'$  will coincide with  $RS$ . § 535

Hence the planes  $R'P'$  and  $RP$  will coincide. § 526

Similarly the planes  $M'S'$  and  $MS$  will coincide.

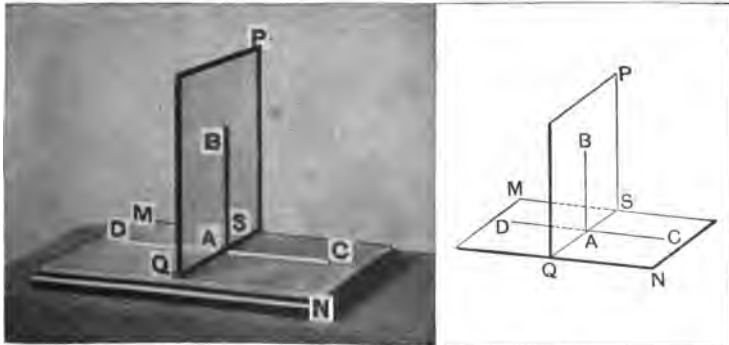
The dihedral angles are therefore equal. § 15  
Q. E. D.

**573. COR.** *Two vertical diedral angles are equal.*



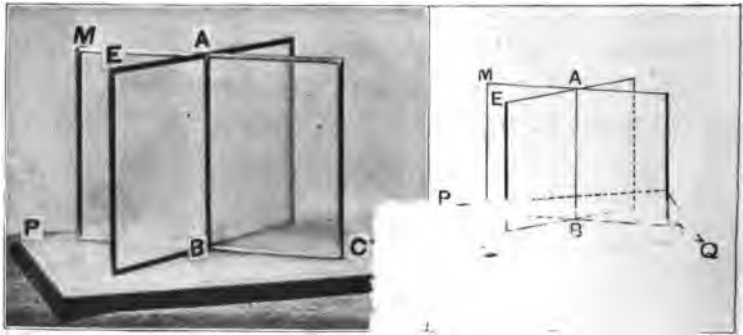
PROPOSITION XVII. THEOREM

**574.** *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the plane.*



GIVEN—the straight line  $AB$  perpendicular to the plane  $MN$  at  $A$ .  
and the plane  $PQ$  passed through  $AB$  intersecting  $MN$  in  $QS$ .

TO PROVE  $PQ$  perpendicular to  $MN$



**GIVEN**—the planes  $MC$  and  $EB$  perpendicular to the line  $AB$  and intersecting in  $AB$ .

**TO PROVE**  $AB$  perpendicular to  $PQ$ .

Through any point of  $AB$  draw a line perpendicular to  $PQ$ .

This line will lie in both  $MC$  and  $EB$ . § 577

It must therefore coincide with the line  $AB$ .

Therefore  $AB$  is perpendicular to  $PQ$ . **Q. E. D.**

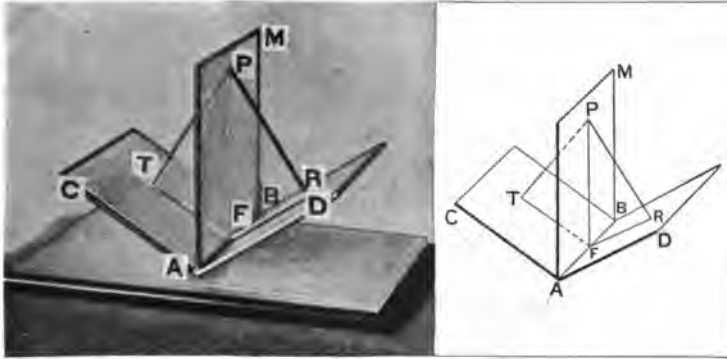
#### PROPOSITION XX. THEOREM

**579.** *Every point in the plane that bisects a dihedral angle is equally distant from the faces of that angle.*

**GIVEN**—the plane  $MA$  bisecting the dihedral angle  $DABC$ . Let  $P$  be any point in  $MA$ , and let  $PT$  and  $PR$  be the perpendiculars dropped from  $P$  to the faces  $BC$  and  $BD$  of the dihedral angle.

**TO PROVE**  $PT = PR$ .

Through  $PT$  and  $PR$  pass a plane intersecting the planes  $BC$ ,  $BD$ , and  $MA$  in  $FT$ ,  $FR$ , and  $FP$  respectively.



Since the line  $PT$  is perpendicular to the plane  $BC$ , the plane  $PRT$  is perpendicular to the plane  $BC$ . § 574

Similarly the plane  $PRT$  is perpendicular to the plane  $BD$ .

Therefore  $PRT$  is perpendicular to their intersection  $AB$ . § 578

Hence  $AB$  is perpendicular to  $FT$ ,  $FP$ , and  $FR$ . § 530

Hence  $PFT$  and  $PFR$  are the plane angles of the equal dihedral angles  $MABC$  and  $MABD$ . § 567

Therefore angle  $PFT = \text{angle } PFR$ . § 571

Consequently the right triangles  $PTF$  and  $PRF$  are equal. § 85

Therefore  $PT = PR$ . Q. E. D.

**580. COR.** *The locus of all points within a dihedral angle equally distant from its faces is the plane which bisects the dihedral angle.*

*Hint.*—It has been proved that all points in the bisecting plane possess the required property. It only remains to prove that any point outside does not, or that any point which possesses the required property must lie in  $AM$ . Let  $P'$  be such a point. Pass a plane through  $P'$  and the edge  $AB$ , and make constructions analogous to those in the preceding figure. Then prove that the plane  $P'AB$  must bisect the dihedral angle.

## PROPOSITION XXI. THEOREM

**581.** *Through any straight line a plane can be passed perpendicular to any plane; and only one such plane can be drawn unless the given line is itself perpendicular to the given plane.*



**GIVEN** the straight line  $AB$  and the plane  $MN$ .

**TO PROVE**—a plane can be drawn through  $AB$  perpendicular to  $MN$ .

From any point  $B$  of  $AB$  drop a perpendicular  $BB'$  to  $MN$ .

The plane passed through  $AB$  and  $BB'$  will be perpendicular to  $MN$ . § 574

Hence *one* plane can be passed through  $AB$  perpendicular to  $MN$ .

Now no other plane can be passed through  $AB$  perpendicular to  $MN$  unless  $AB$  is perpendicular to  $MN$ .

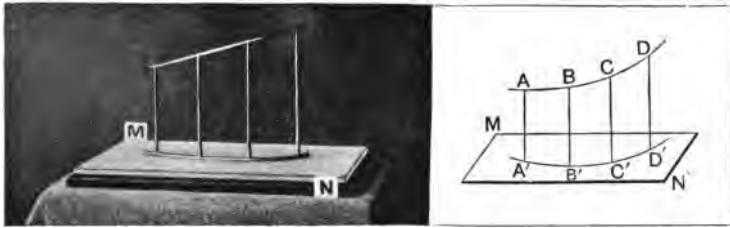
For this other plane would contain  $BB'$ . § 577

And would coincide with the first plane, since both contain the intersecting lines  $AB$  and  $BB'$ . § 526 III

Q. E. D.

**582. Def.**—The **projection** of a point upon a plane is the foot of the perpendicular drawn from the point to the plane.

Thus  $A'$  is the projection of the point  $A$  upon the plane  $MN$ .

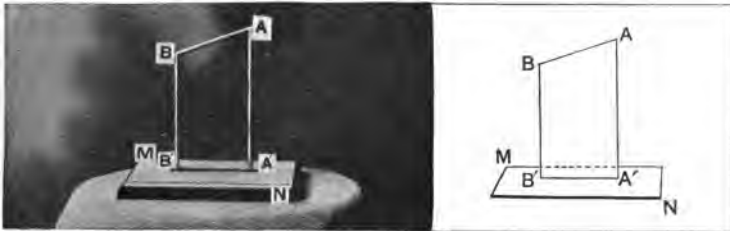


**583. Def.**—The projection of a line upon a plane is the locus of the projections of its points.

Thus  $A'B'C'D'$  is the projection of  $ABCD$  upon  $MN$ .

PROPOSITION XXII. THEOREM

**584.** *The projection of a straight line upon a plane is a straight line.*



**GIVEN**—the projection  $A'B'$  of the straight line  $AB$  upon the plane  $MN$ .

**TO PROVE**  $A'B'$  a straight line.

Through  $AB$  pass a plane perpendicular to  $MN$ .

The perpendiculars drawn from the various points of  $AB$  to  $MN$  must lie in this plane. § 577

Hence their feet will lie in the intersection of  $MN$  with this plane.

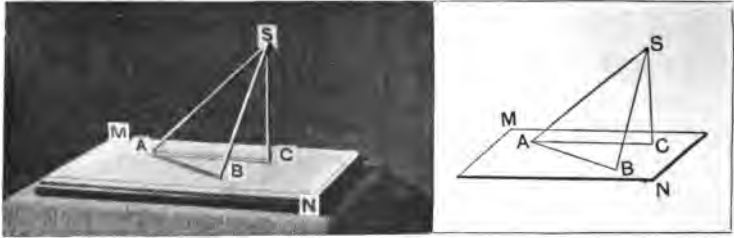
Therefore  $A'B'$  must be the intersection and is a straight line. § 528

B

Q. E. D.

## PROPOSITION XXIII. THEOREM

**585.** *The acute angle which a straight line makes with its own projection upon a plane is the least angle which it makes with any line in that plane.*



**GIVEN**—the straight line  $AS$ , its projection  $AC$  upon the plane  $MN$ , and  $AB$  any other straight line in  $MN$  through  $A$ .

**TO PROVE**                      angle  $SAC <$  angle  $SAB$ .

Take  $AB = AC$  and draw  $SC$  and  $SB$ .

Then the triangles  $SAC$  and  $SAB$  have two sides of one equal respectively to two sides of the other.

But the third side  $SC$  of one is less than the third side  $SB$  of the other.

§ 536

Therefore                      angle  $SAC <$  angle  $SAB$ .

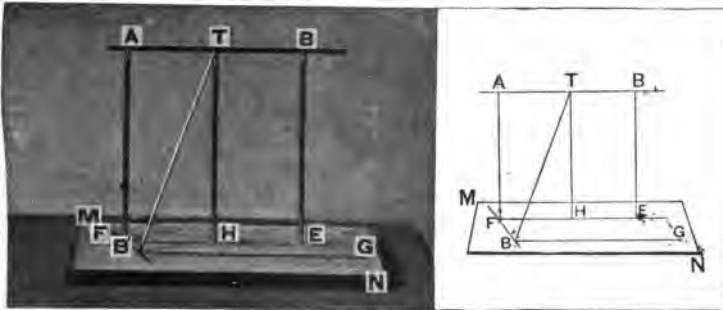
§ 93

Q. E. D.

**586. Def.**—The acute angle which a straight line makes with its own projection upon a plane is called the **inclination of the line to the plane**.

## PROPOSITION XXIV. THEOREM

**587.** *Between two straight lines not in the same plane a common perpendicular can be drawn, and only one.*



GIVEN— $AB$  and  $FB'$ , two straight lines not in the same plane.

TO PROVE—that a common perpendicular can be drawn between them, and only one.

Through any point  $B'$  of  $FB'$  draw a line  $B'G$  parallel to  $AB$  and let  $MN$  be the plane containing  $FB'$  and  $B'G$ .

$MN$  is parallel to  $AB$ . § 550

Pass a plane through  $AB$  perpendicular to the plane  $MN$ , intersecting  $FB'$  at  $F$  and  $MN$  in  $FE$ ,

$FE$  is parallel to  $AB$ . § 546

At  $F$  erect a perpendicular  $FA$  to  $FE$  in the plane  $FB$ , hence perpendicular to  $MN$  and to  $FB'$ . § 575

Since  $FA$  is perpendicular to  $FE$ , it is perpendicular to  $AB$ . § 36

Therefore  $FA$  is a common perpendicular to  $AB$  and  $FB'$ .

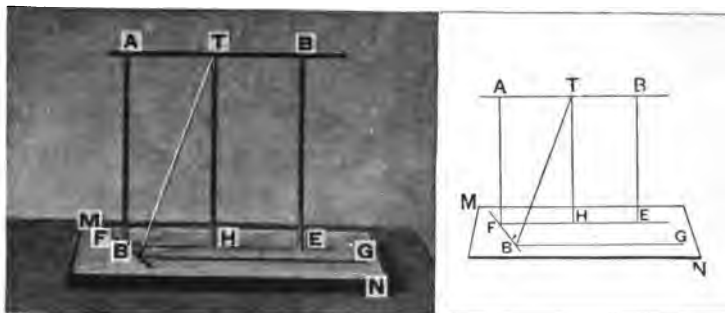
No other line as  $TB'$  can be perpendicular to both  $AB$  and  $FB'$ .

For  $TB'$  would also be perpendicular to  $B'G$ , parallel to  $AB$ .

Hence  $TB'$  would be perpendicular to  $MN$ . § 531

But  $TH$  drawn in  $AE$  perpendicular to  $FE$  is perpendicular to  $MN$ . § 575





Hence there would be two perpendiculars from  $T$  to  $MN$ , which is impossible. § 537

Therefore  $TB'$  cannot be perpendicular to both  $AB$  and  $FB'$ . Q. E. D.

#### POLYEDRAL ANGLES

**588.** *Def.*—When three or more planes meet in a point, they are said to form a **polyedral angle**.



Thus the planes  $AOB$ ,  $BOC$ ,  $COD$ ,  $DOA$  passing through the common point  $O$  form the polyedral angle  $O - ABCD$ .

The common point  $O$  is called the **vertex** of the polyedral angle; the planes  $AOB$ ,  $BOC$ , etc., are called the **faces**; the intersections  $OA$ ,  $OB$ , etc., of the faces are called

the **edges**; the angles  $AOB$ ,  $BOC$ , etc., are called the **face angles** of the polyedral angle.

The faces of a polyedral angle are supposed to be indefinite in extent.

In order to show clearly in a figure the relative position of the edges, they are represented as being cut by a plane, as  $AC$ .

**589. Def.**—The polygon formed by the intersection of a plane with the faces of a polyedral angle is called a **section** of the polyedral angle.

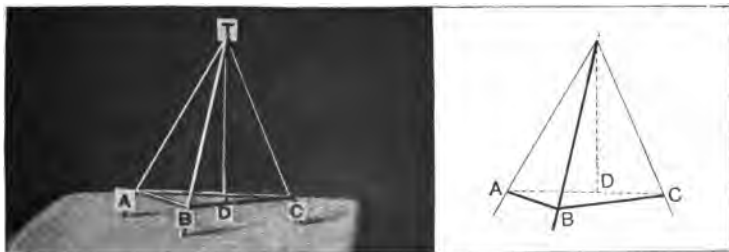
**590. Def.**—A polyedral angle is **convex** when any section by a plane forms a convex polygon.

**591. Def.**—The dihedral angles formed by the faces, together with the face angles, are called the **parts** of the polyedral angle.

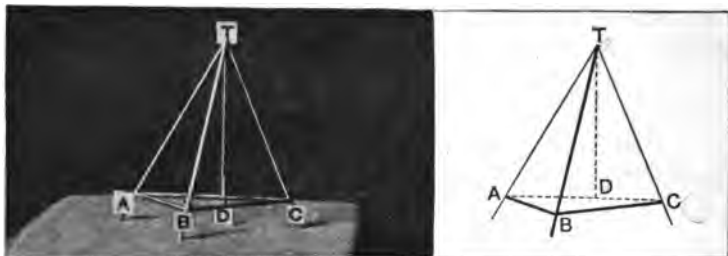
**592. Def.**—A polyedral angle of three faces is called a **triedral angle**.

PROPOSITION XXV. THEOREM

**593.** *The sum of any two face angles of a triedral angle is greater than the third.*



The theorem requires proof only when the third angle is greater than each of the others.



GIVEN—the trihedral angle  $T-ABC$  in which the face angle  $ATC$  is greater than either  $ATB$  or  $BTC$ .

TO PROVE  $ATB + BTC > ATC$ .

In the face  $ATC$  draw  $TD$ , making the angle  $ATD = ATB$ .

Take  $TB = TD$ , and through  $B$  and  $D$  pass a plane cutting the three faces in  $AB$ ,  $BC$ , and  $AC$ .

The triangles  $ATB$  and  $ATD$  are equal. § 79

Hence  $AB = AD$ .

But  $AB + BC > AC$ .

By subtraction  $BC > DC$ .

The triangles  $BTC$  and  $DTC$  have two sides of one equal to two sides of the other, and the third side  $BC$  of one greater than the third side  $DC$  of the other.

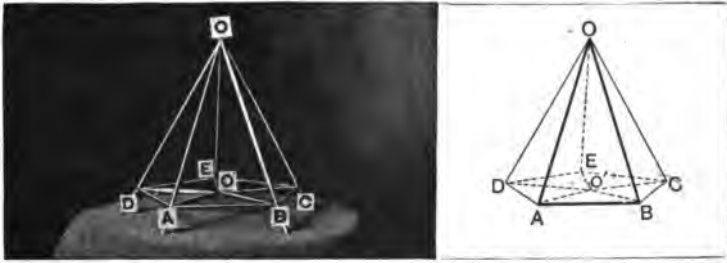
Therefore  $BTC > DTC$ . § 93

By construction  $ATB = ATD$ .

Adding  $ATB + BTC > ATC$ . Q. E. D.

#### PROPOSITION XXVI. THEOREM

**594.** *The sum of the face angles of any convex polyedral angle is less than four right angles.*



GIVEN the convex polyhedral angle  $O-ABCED$ .

TO PROVE  $AOB + BOC + \text{etc.} < \text{four right angles}$ .

The section  $ABCED$  of the polyhedral angle is a convex polygon. § 590

Join any point  $O'$  in this polygon to its vertices.

In the trihedral angle  $A$  we have

$$OAD + OAB > DAB, \quad \text{§ 593}$$

Similarly  $OBA + OBC > ABC$ , etc.

Adding these inequalities we get :

The sum of the base angles of the triangles about  $O >$  the sum of the base angles of the triangles about  $O'$ .

But the sum of all the angles of the triangles about  $O =$  the sum of all the angles of the triangles about  $O'$ .

[There being the same number of triangles having  $O$  for vertex as having  $O'$ , and each triangle containing two right angles.]

Subtracting the inequality from the equality we get :

Sum of the angles whose vertex is  $O <$  sum of the angles whose vertex is  $O'$ .

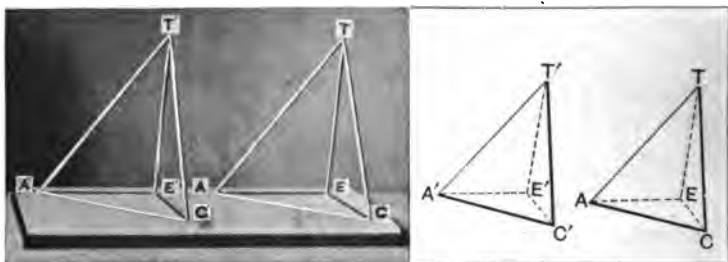
But the sum of the angles at  $O'$  is four right angles. § 28

Therefore the sum of the angles at  $O$  is less than four right angles.

Q. E. D.

## PROPOSITION XXVII. THEOREM

**595.** *Two triedral angles are equal, if two face angles and the included dihedral angle of one are respectively equal to two face angles and the included dihedral angle of the other, the parts given equal being arranged in the same order.*



**GIVEN**—the triedral angles  $T-ACE$  and  $T'-A'C'E'$  having angle  $CTA = \text{angle } C'T'A'$ ; angle  $ETA = \text{angle } E'T'A'$ ; dihedral angle  $TA = \text{dihedral angle } T'A'$ ; the parts given equal being arranged in the same order.

**TO PROVE**  $T-ACE = T'-A'C'E'$ .

Place the triedral angles so that the equal dihedral angles  $T'A'$  and  $TA$  shall coincide, the point  $T'$  falling on  $T$ .

The angles  $C'T'A'$  and  $CTA$  will then lie in the same plane.

Since they are equal,  $T'C'$  will coincide with  $TC$ .

Similarly  $T'E'$  will coincide with  $TE$ .

Then the third faces  $TEC$  and  $T'E'C'$  will coincide.

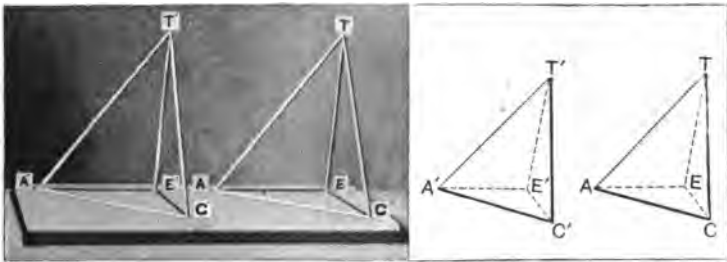
§ 526 III

Therefore the triedral angles coincide throughout and are equal.

Q. E. D.

PROPOSITION XXVIII. THEOREM

**596.** *Two triedral angles are equal, if two diedral angles and the included face angle of one are respectively equal to two diedral angles and the included face angle of the other, the parts given equal being arranged in the same order.*



**GIVEN**—the triedral angles  $T-ACE$  and  $T'-A'C'E'$  having face angle  $CTA =$  face angle  $C'T'A'$ ; diedral angle  $TC =$  diedral angle  $T'C'$ ; diedral angle  $TA =$  diedral angle  $T'A'$ ; the parts given equal being arranged in the same order.

**TO PROVE**  $T-ACE = T'-A'C'E'$ .

Place the triedral angles so that the equal angles  $C'T'A'$  and  $CTA$  shall coincide.

Since diedral angle  $T'C' =$  diedral angle  $TC$ , the plane  $TCE$  will take the direction of  $T'C'E'$ .

Similarly the plane  $TAE$  will take the direction of  $T'A'E'$ .

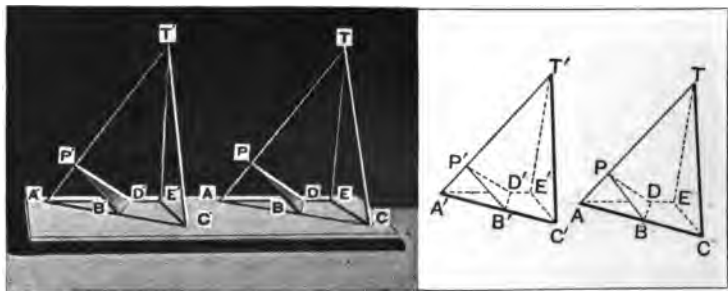
Then the intersection  $TE$  must lie somewhere in the plane  $T'E'C'$  and somewhere in  $T'E'A'$ , and therefore must coincide with the intersection  $T'E'$ .

Therefore the triedral angles coincide throughout and are equal.

Q. E. D.

## PROPOSITION XXIX. THEOREM

**597.** *Two triedral angles are equal, if the three face angles of one are respectively equal to the three face angles of the other, provided the equal face angles are arranged in the same order.*



**GIVEN**—the triedral angles  $T-ACE$  and  $T-A'CE'$  having face angle  $ATC = \text{face angle } A'TC'$ ; face angle  $CTE = \text{face angle } C'T'E'$ ; face angle  $ETA = \text{face angle } E'T'A'$ ; the equal face angles being arranged in the same order.

**TO PROVE**  $T-ACE = T-A'CE'$ .

On the six edges take  $TA = TC = TE = T'A' = T'C' = T'E'$ , and join  $AC, CE, EA, A'C', C'E', E'A'$ .

The triangles  $ATC$  and  $A'T'C'$  are equal. § 79

Hence their homologous sides  $AC$  and  $A'C'$  are equal.

Similarly  $CE = C'E'$  and  $EA = E'A'$ .

Therefore the triangles  $ACE$  and  $A'C'E'$  are equal. § 89

At any point  $P$  in  $TA$  draw  $PB$  in the face  $ATC$  and  $PD$  in the face  $ATE$  perpendicular to  $TA$ .

$PB$  must meet  $AC$ .

[For if  $PB$  were parallel to  $CA$ ,  $CA$  would be perpendicular to  $TA$  (§ 36), which cannot be the case, since the angle  $TAC$  is acute, being a base angle of an isosceles triangle.]

And  $PB$  must meet  $AC$  upon that side of  $TA$  on which  $C$  lies.

[For if it met  $AC$  on the other side, there would be formed a triangle such that the sum of two of its angles, those at  $P$  and  $A$ , would be greater than two right angles, which is impossible.]

Likewise  $PD$  must meet  $EA$  on that side of  $TA$  on which  $E$  lies.

Let the points of meeting be  $B$  and  $D$ . Join  $BD$ .

On the edge  $T'A'$  take  $A'P' = AP$ , and at  $P'$  repeat the same construction in the triedral angle  $T'$ .

The right triangles  $APB$  and  $A'P'B'$  are equal. § 86

[Having a side,  $PA$ , and acute angle  $PAB$  of one equal to a side and homologous acute angle of the other.]

Therefore the homologous sides  $AB$  and  $A'B'$  and  $PB$  and  $P'B'$  are respectively equal.

Similarly  $AD = A'D'$  and  $PD = P'D'$ .

Next, the triangles  $BAD$  and  $B'A'D'$  are equal. § 79

[Having two sides  $AB$  and  $AD$  and the included angle  $DAB$  of one equal to two sides and the included angle of the other.]

Hence  $BD = B'D'$ .

Finally, the triangles  $PBD$  and  $P'B'D'$  are equal. § 89

Therefore the homologous angles  $BPD$  and  $B'P'D'$ , that is, the plane angles of the diedral angles  $TA$  and  $T'A'$ , are equal.

Therefore the diedral angles  $AT$  and  $A'T'$  are also equal.

§ 572

Therefore the triedral angles  $T-ACE$  and  $T'-A'C'E'$  are equal.

§ 595

Q. E. D.

**598.** *Outline of steps* used in the last proposition:

- I. Proof that the *large face* triangles, viz.,  $TAC$  and  $T'A'C'$ , etc., are equal.
- II. Proof that the *large base* triangles, viz.,  $ACE$  and  $A'C'E'$ , etc., are equal.
- III. Proof that the *small face* triangles, viz.,  $APB$  and  $A'P'B'$ , etc., are equal.
- IV. Proof that the *small base* triangles, viz.,  $ABD$  and  $A'B'D'$ , etc., are equal.
- V. Proof that the triangles  $PBD$  and  $P'B'D'$  are equal.



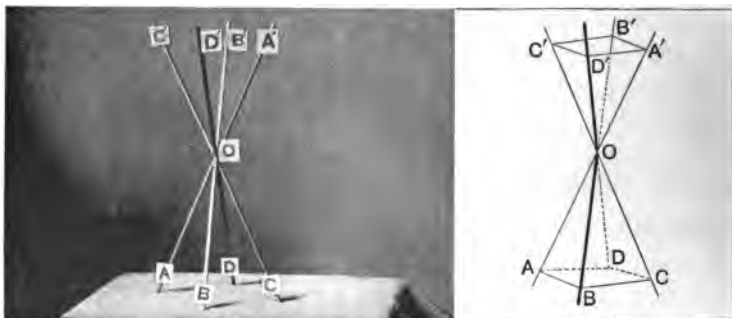
**599. Def.**—Two polyedral angles are **vertical**, if the edges of one are the prolongations, through the vertex, of the edges of the other.

**600. Def.**—Two polyedral angles are **symmetrical**, if all the parts of one are equal to those of the other but arranged in opposite order.

Symmetrical polyedral angles are not in general equal, that is, cannot be made to coincide, just as we cannot put a right glove on the left hand.

PROPOSITION XXX. THEOREM

**601.** *Two vertical polyedral angles are symmetrical.*



**GIVEN**—the vertical polyedral angles  $O-ABCD$  and  $O-A'B'C'D'$ .

**TO PROVE** them symmetrical.

The lines  $OA'$ ,  $OB'$ , etc., are the prolongations of the lines  $OA$ ,  $OB$ , etc., respectively.

Therefore the angles  $A'OB'$ ,  $B'OC'$ , etc., are equal respectively to the angles  $AOB$ ,  $BOC$ , etc. § 30

The planes  $A'OB'$ ,  $B'OC'$ , etc., are the prolongations of the planes  $AOB$ ,  $BOC$ , etc., respectively. § 526 III

Hence the dihedral angles  $OA', OB'$ , etc., are equal respectively to the dihedral angles  $OA, OB$ , etc. § 573

[Vertical dihedral angles are equal.]

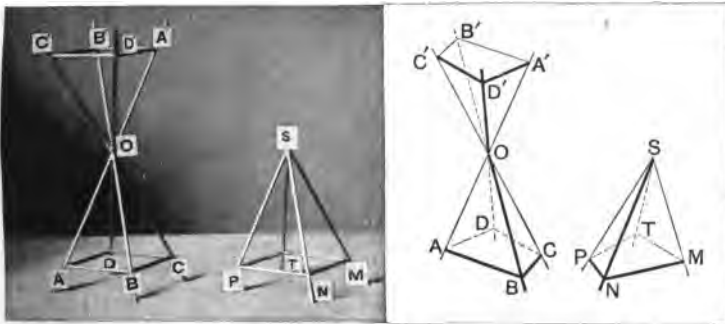
But the equal parts of the two polyedral angles are arranged in opposite order.\*

Therefore they are symmetrical. § 600

[Having all the parts of one equal to those of the other, but arranged in opposite order.] Q. E. D.

PROPOSITION XXXI. THEOREM

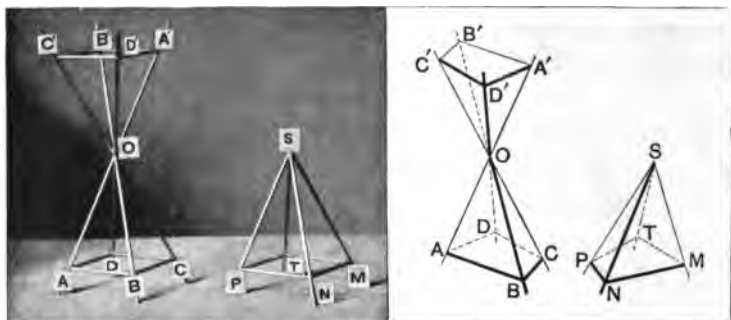
**602.** *Either of two symmetrical polyedral angles is equal to the vertical of the other.*



GIVEN—two symmetrical polyedral angles,  $O-ABCD$  and  $S-MNPT$ , the points  $M, N, P, T$ , corresponding to the points  $A, B, C, D$ .

TO PROVE—that  $S-MNPT$  can be made to coincide with  $O-A'B'C'D'$ , the vertical of  $O-ABCD$ .

\* A convenient way of seeing this is to conceive the eye placed at  $O$ . Then, if we look at the points  $A'B'C'D'$ , we find that they follow each other in an order of rotation in the same direction as the hand of a clock moves. This order is called "clockwise." But if we look at  $ABCD$ , still keeping the eye at  $O$ , the order  $ABCD$  is "counter-clockwise."



The parts of  $S-MNPT$  and  $O-ABCD$  are equal each to each and arranged in opposite order. § 600

[Two symmetrical polyhedral angles have their parts equal each to each and arranged in opposite order.]

Also the parts of  $O-A'B'C'D'$  and  $O-ABCD$  are equal each to each and arranged in opposite order. § 601

[Two vertical polyhedral angles are symmetrical.]

Therefore the parts of  $S-MNPT$  and  $O-A'B'C'D'$  are equal each to each and arranged in the same order.

Place the polyhedral angle  $S-MNPT$  so that its dihedral angle  $SM$  shall coincide with the equal dihedral angle  $OA'$ , the point  $S$  falling at  $O$ .

Since the parts of the two polyhedral angles are arranged in the same order, the angles  $NSM$  and  $B'OA'$  will then lie in the same plane.

Since they are equal,  $SN$  will coincide with  $OB'$ .

Similarly we can show that the next edge  $SP$  will coincide with  $OC'$  and so on until all the edges and therefore all the faces coincide.

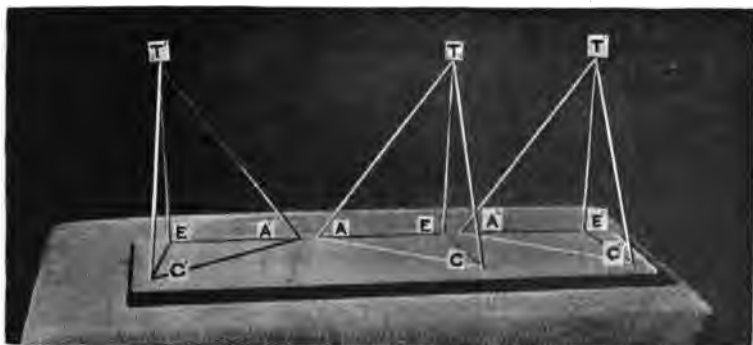
Hence the polyhedral angles  $S-MNPT$  and  $O-A'B'C'D'$  coincide and are equal.

Q. E. D.

PROPOSITION XXXII. THEOREM

**603.** *Two triedral angles are symmetrical :*

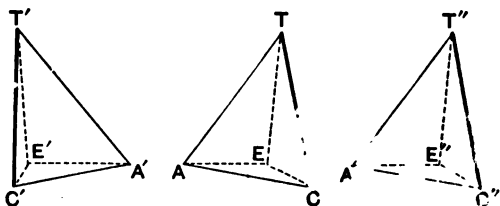
- I. *If two face angles and the included dihedral angle of one are respectively equal to two face angles and the included dihedral angle of the other.*
- II. *If two dihedral angles and the included face angle of one are respectively equal to two dihedral angles and the included face angle of the other.*
- III. *If the three face angles of one are respectively equal to the three face angles of the other.*  
*Provided the parts given equal are arranged in opposite order.*



Let  $T-ACE$  (or  $T$ ) and  $T'-A'C'E'$  (or  $T'$ ) be the two triedral angles, the parts given equal being arranged in opposite order.

Also let  $T''$  be a triedral angle symmetrical to  $T'$ , that is, having corresponding parts equal, but arranged in opposite order.

Therefore  $T''$  and  $T$  have parts equal each to each and arranged in the same order.



Therefore in either of the three cases,  $T'$  is equal to  $T''$ .

But  $T''$  was constructed symmetrical to  $T'$ .

Therefore  $T$ , which equals  $T''$ , is also symmetrical to  $T'$ .

**604. Def.**—A trihedral angle is **isosceles**, if two of its face angles are equal.

**605. Exercise.**—If one of two symmetrical trihedral angles is isosceles, the other is also, and the two can be made to coincide and are equal.

It will be noted, however, that the parts which correspond by symmetry will not be the ones which coincide.

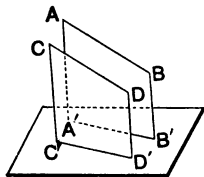
## PROJECTIONS

**606. Exercise.**—The projections on a plane of parallel lines are parallel.

*Hint.*—Prove first that the projecting planes are parallel, using § 551.

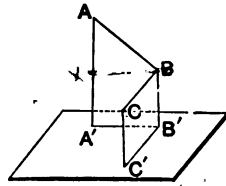
This principle is of great importance in the theory of shades and shadows.

It is not true in general that if two lines make an angle with each other, their projections on a plane will make the same angle.

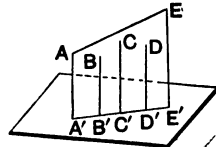


**607. Exercise.**—The projection on a plane of a right angle is a right angle provided one of the sides is parallel to the plane.

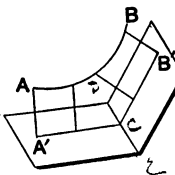
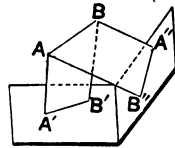
*Hint.*—Prove first that the side which is parallel to the plane is perpendicular to the projecting plane of the other; then that the two projecting planes are perpendicular, and, finally, that the projections of the sides are perpendicular.



**608. Exercise.**—If the projections on a plane of a number of points lie in a straight line, the points must lie in a plane. § 576

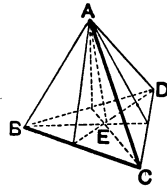


**609. Exercise.**—If the projections of a line on each of two intersecting planes be straight, the line itself must be straight except in one case. State that case.

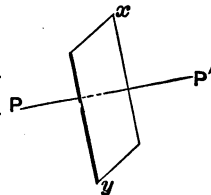


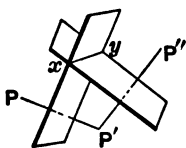
LOCI

**610. Exercise.**—In any triedral angle the three planes bisecting the three dihedral angles intersect in a common straight line, which is the locus of points within the triedral angle equidistant from its faces.

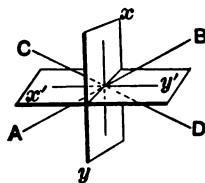


**611. Exercise.**—Find, and prove correct, the locus of all points in space equidistant from two given points.





**612. Exercise.**—Find, and prove correct, the locus of all points equidistant from three given points.



**613. Exercise.**—The locus of points equidistant from two intersecting straight lines is the pair of planes passed through the bisectors of the angles formed by the lines and perpendicular to the plane of the lines.

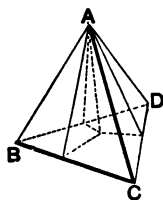
*Hint.*—Apply §§ 595, 86.

**614. Exercise.**—Find the locus of points at a given distance from a given plane.

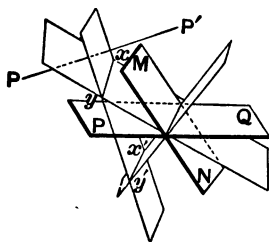
**615. Exercise.**—Find the locus of points equidistant from two parallel planes.

**616. Exercise.**—Find the locus of points equidistant from two intersecting planes.

**617. Exercise.**—Find the locus of points equidistant from three intersecting straight lines not in the same plane.

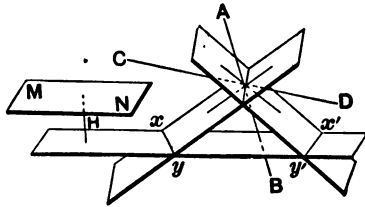


**618. Exercise.**—In any trihedral angle the three planes passed through the bisectors of the three face angles, and perpendicular to these faces respectively, intersect in a common straight line, every point of which is equidistant from the edges of the trihedral angle.



**619. Exercise.**—Find, and prove correct, the locus of points which are equidistant from two given planes, and at the same time equidistant from two given points.

**620. Exercise.** — Find, and prove correct, the locus of points at a given distance from a given plane, and at the same time equidistant from two intersecting straight lines.



Does the figure show all the lines of the locus?

PROBLEMS OF CONSTRUCTION

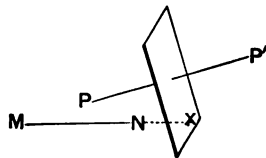
The constructions of solid geometry differ from those of plane geometry in that we cannot perform them with ruler and compasses, or with any instruments of drawing.

We shall therefore consider a problem of construction in solid geometry solved when it is reduced to one or more of the following elementary constructions which we assume can be performed, viz. :

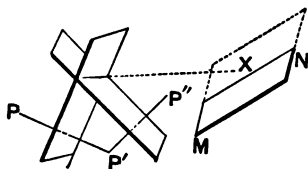
- (1.) A plane can be drawn through any three given points.
- (2.) The intersection of a plane with any given straight line or with any given plane can be determined.
- (3.) A straight line can be drawn through any given point perpendicular to any given plane.

**621. Exercise.** — Determine a point in a given straight line which shall be equidistant from two given points in space.

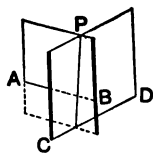
Do not assume that the given line and the given points are in the same plane, and avoid similar assumptions in the following exercises.



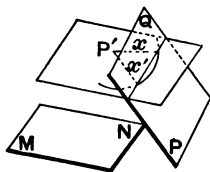




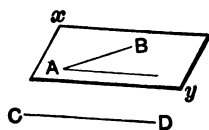
**622. Exercise.**—Determine a point in a plane  $MN$  which shall be equidistant from three given points in space,  $P$ ,  $P'$ , and  $P''$ .



**623. Exercise.**—Through a given point  $P$  in space determine a straight line which shall cut two given straight lines  $AB$  and  $CD$ .

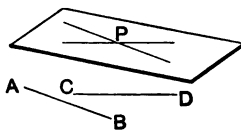


**624. Exercise.**—Given a point  $P'$  and any two non-parallel planes  $MN$  and  $PQ$ . From the point draw a straight line of given length terminating in one of the planes and parallel to the other.



**625. Exercise.**—Show how to pass a plane through a straight line  $AB$  parallel to another straight line  $CD$ .

*Hint.*—Apply § 550.



**626. Exercise.**—Show how to pass a plane through a point  $P$  parallel to two given straight lines  $AB$  and  $CD$ .

# GEOMETRY OF SPACE

## BOOK VII

### POLYEDRONS

**627.** *Def.*—A **polyedron** is a geometrical solid bounded by planes.

The intersections of the bounding planes are called the **edges** of the polyedron; the intersections of the edges are called the **vertices**; the portions of the bounding planes bounded by the edges are called the **faces**.

The least number of faces that a polyedron can have is four; for three planes by intersecting form a triedral angle, and one more plane is necessary to enclose with these a definite portion of space.

**628.** *Def.*—A polyedron of four faces is called a **tetraedron**; one of six faces, a **hexaedron**; one of eight faces, an **octaedron**; one of twelve faces, a **dodecaedron**; one of twenty faces, an **icosaedron**.



ICOSAEDRON

DODECAEDRON

OCTAEDRON

HEXAEDRON

TETRAEDRON

20

12

8

6

4

**629. Def.**—A polyedron is **convex** when no face, if produced, will enter the polyedron.

All the polyedrons treated of in this book will be understood to be convex.

#### PRISMS. PARALLELOPIPEDS

**630. Defs.**—A **prismatic surface** is a surface composed of planes passed between each successive pair of a system of parallel lines.



The parallel lines are called the **edges** of the prismatic surface.

#### PROPOSITION I. THEOREM

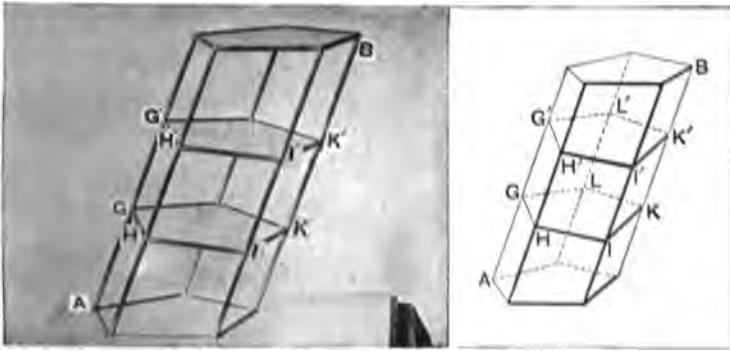
**631.** *The sections of a prismatic surface made by two parallel planes cutting its edges are equal polygons.*

**GIVEN**—the prismatic surface  $AB$  cut by two parallel planes in the sections  $GHIKL$  and  $G'H'I'K'L'$ .

**TO PROVE** these polygons are equal.

The sides  $GH$ ,  $HI$ , etc., are parallel respectively to  $G'H'$ ,  $H'I'$ , etc. § 544

Hence  $GH = G'H'$ ,  $HI = H'I'$ . § 118



Also  $\text{angle } GHI = \text{angle } G'H'I'$ ,  
 $\text{angle } HIK = \text{angle } H'I'K'$ , etc. § 557

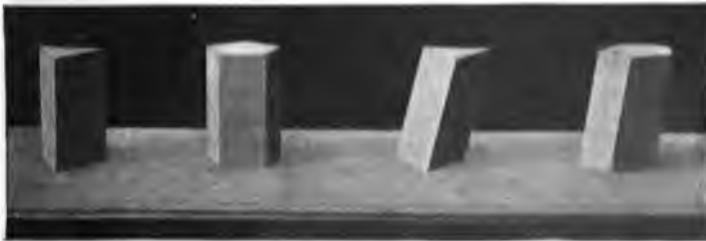
The polygons  $GHIKL$  and  $G'H'I'K'L'$  are therefore mutually equilateral and equiangular.

Hence they can be made to coincide and are equal. Q. E. D.

**632. COR.** A prismatic surface can be generated by a straight line moving so as to remain always parallel to a fixed straight line (drawn parallel to the edges) and always cutting the perimeter of a section.

*Hint.*—By plane geometry a straight line can move across each face remaining parallel to the lateral edges.

**633. Defs.**—A prism is a polyedron bounded by a prismatic surface and two parallel planes.



PRISMS

The equal sections of the prismatic surface formed by the parallel planes are called the **bases** of the prism; the portion of the prismatic surface between the bases consists of the **lateral faces**; the portions of the edges of the prismatic surface between the bases are the **lateral edges** of the prism.

**634. Defs.**—A **right prism** is one whose lateral edges are perpendicular to its bases.

An **oblique prism** is one whose lateral edges are not perpendicular to its bases.

**635. Def.**—A **regular prism** is one whose bases are regular polygons and whose lateral edges are perpendicular to its bases.



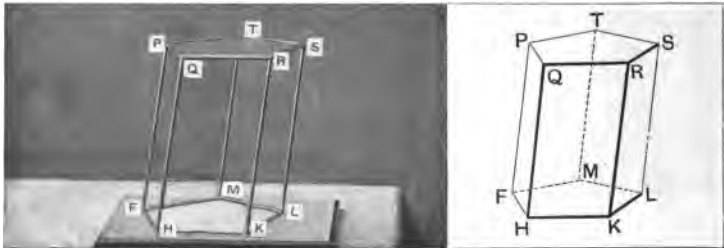
RIGHT PRISM

REGULAR PRISM

OBLIQUE PRISMS

#### PROPOSITION II. THEOREM

**636.** *The lateral faces of a prism are parallelograms.*



GIVEN the prism  $FS$ .

TO PROVE its lateral faces are parallelograms.

Consider the lateral face  $FQ$ .

Its sides  $FP$  and  $HQ$  are parallel, being edges of the prismatic surface. § 630

Also  $FH$  and  $PQ$  are parallel, being the intersections of two parallel planes with a third. § 544

Therefore  $FQ$  is a parallelogram. § 114

Similarly the other lateral faces are proved to be parallelograms. Q. E. D.

**637.** COR. I. *The lateral edges of a prism are equal.*

**638.** COR. II. *The lateral faces of a right prism are rectangles.*

**639.** Def.—A **parallelepiped** is a prism whose bases are parallelograms.

**640.** Def.—A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to its bases.



OBLIQUE  
PARALLELOPIPED

RIGHT  
PARALLELOPIPED

RECTANGULAR  
PARALLELOPIPED

CUBE

**641.** Def.—A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.

**642.** Def.—A **cube** is a right parallelepiped whose bases are squares and whose lateral edges are equal to the sides of its base.

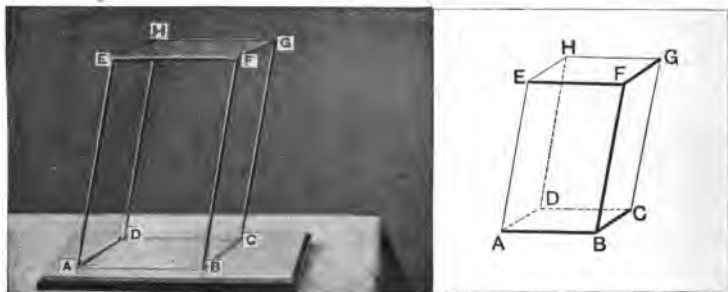
**643.** COR. III. *All the faces of a parallelepiped are parallelograms.*

**644.** COR. IV. *All the faces of a rectangular parallelepiped are rectangles.*

**645.** COR. V. *All the faces of a cube are equal squares.*

PROPOSITION III. THEOREM

**646.** *Any two opposite faces of a parallelepiped may be taken as its bases.*



**GIVEN**—the parallelepiped  $AG$ , the bases being first taken as  $AC$  and  $EG$ .

**TO PROVE**—that any other two opposite faces, as  $AF$  and  $DG$ , may be taken as bases.

The four lines  $AD, BC, FG, EH$  are parallel to each other.

§§ 114, 549

They may therefore be taken as the edges of a prismatic surface.

§ 630

Also  $AB$  and  $AE$  are parallel to  $DC$  and  $DH$  respectively.

§ 114

Hence the planes  $AF$  and  $DG$  are parallel.

§ 551

Therefore the parallelepiped may be considered a prism having  $AF$  and  $DG$  as bases.

§ 633

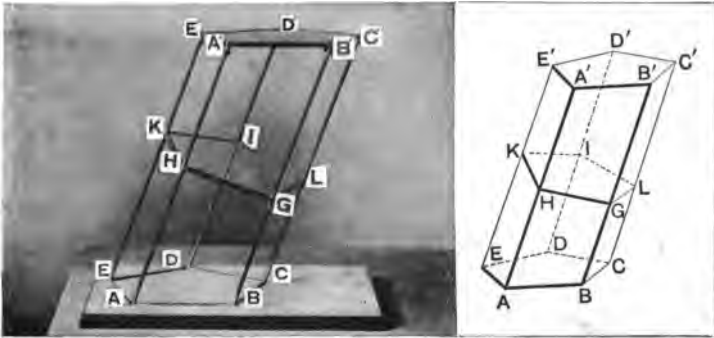
Q. E. D

**647. Def.**—A **right section** of a prism is the section formed by a plane perpendicular to the lateral edges.

**648. Def.**—The **lateral area** of a prism is the sum of the areas of its lateral faces. ✓

PROPOSITION IV. THEOREM

**649.** *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*



**GIVEN**—the prism  $AC'$ , of which  $HGLIK$  is a right section.

**TO PROVE**—its lateral area =  $(HG + GL + \text{etc.}) \times AA'$ .

The lateral area consists of the areas of the lateral faces, which are parallelograms. § 636

The area of each parallelogram is its base multiplied by its altitude. § 385

Their bases  $AA'$ ,  $BB'$ , etc., are all equal. § 637

Their altitudes are the lines  $HG$ ,  $GL$ , etc. § 530

Hence by addition we have

$$\text{lateral area} = (HG + GL + \text{etc.}) \times AA'. \quad \text{Q. E. D.}$$

**650. Def.**—The **altitude** of a prism is the perpendicular distance between the planes of its bases.



**651. COR.** *The lateral area of a right prism is equal to the product of the perimeter of its base and its altitude.*

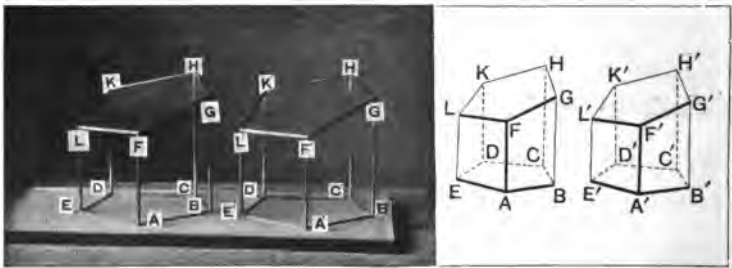
**652. Defs.**—A truncated prism is a polyedron bounded by a prismatic surface and two non-parallel planes.

The sections of the prismatic surface formed by the non-parallel planes are called the **bases** of the truncated prism.

**653. Def.**—A truncated prism is **right** when one of its bases is perpendicular to the lateral edges.

PROPOSITION V. THEOREM

**654.** *Two right truncated prisms are equal, if three lateral edges of one are equal to three corresponding edges of the other and the bases to which they are respectively perpendicular are equal.*



**GIVEN**—the truncated right prisms  $AK$  and  $A'K'$ , having the lateral edges  $AF$  and  $A'F'$ ,  $BG$  and  $B'G'$ ,  $CH$  and  $C'H'$  respectively equal and perpendicular to the equal bases  $ABCDE$ ,  $A'B'C'D'E'$ .

**TO PROVE** that  $AK$  and  $A'K'$  are equal.

Superpose the truncated prisms so that the bases  $ABCDE$  and  $A'B'C'D'E'$  shall coincide.

Then the indefinite lines  $AF$ ,  $BG$ , etc., will coincide respectively with  $A'F'$ ,  $B'G'$ , etc.

Hence the indefinite prismatic surfaces coincide. § 526 IV  
 Since  $AF=A'F'$ ,  $F$  falls on  $F'$ . Similarly  $G$  falls on  $G'$   
 and  $H$  upon  $H'$ .

Hence the planes of the upper bases coincide. § 526 I  
 Therefore the truncated prisms coincide and are equal.

Q. E. D.

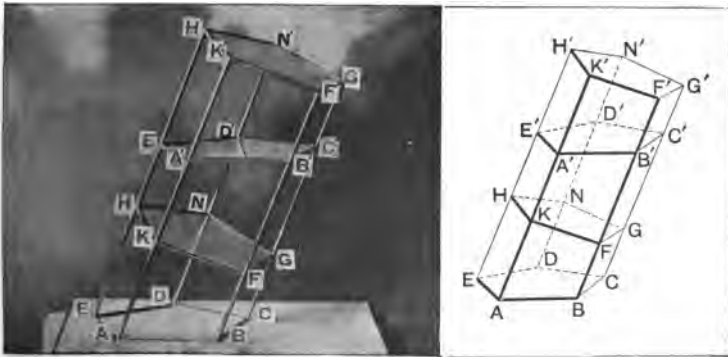
**655. COR.** *Two right prisms are equal, if they have equal bases and equal altitudes.*

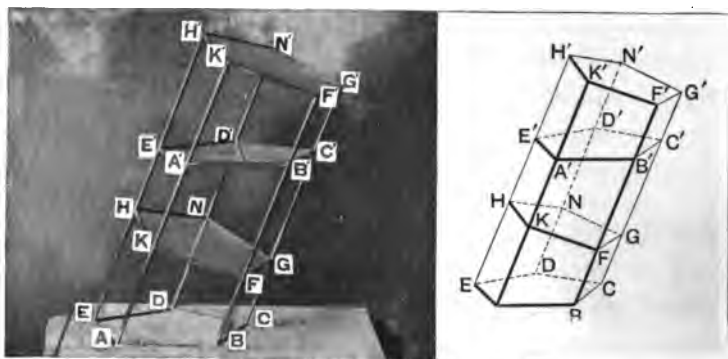
**656. Defs.**—The volume of any solid is its ratio to another solid taken arbitrarily and called the unit of volume.

**657. Def.**—Two solids are equivalent when their volumes are equal.

PROPOSITION VI. THEOREM

**658.** *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.*





**GIVEN**—the oblique prism  $ABCDE-A'$  of which  $KFGNH$  is a section.

Produce  $AA'$  to  $K'$ , making  $KK' = AA'$ , and take a plane parallel to  $KFGNH$ , cutting the other edges at  $G', N', H'$ .

**TO PROVE**—the oblique prism  $ABCDE-A'$  is equal to the right prism  $KFGNH-K'$ .

The truncated right prisms  $AG$  and  $A'G'N'H'$ ,  $KFGNH$  and  $K'F'G'N'H'$  equal.

Also the lateral edges  $AK$ ,  $BF$ , and  $CG$  are equal to  $A'K'$ ,  $B'F'$ , and  $C'G'$ .

Therefore these truncated prisms are equal.

If we take the upper truncated prism  $A'G'N'H'$  from the whole figure, we have left the oblique prism.

If we take the lower truncated prism  $AG$  from the whole figure, we have left the right prism.

Therefore the oblique prism is equivalent to the right prism.

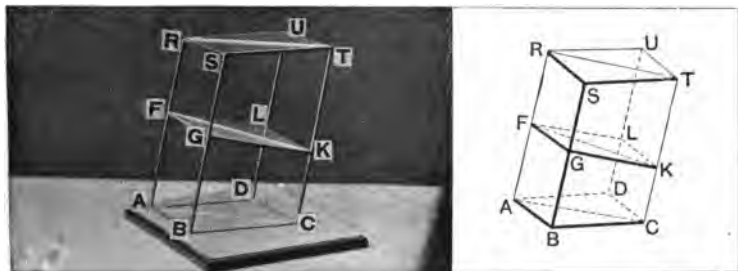
Ax. 3

Q. E. D.

**659. Defs.**—A triangular prism is one whose base is a triangle; a quadrangular, one whose base is a quadrilateral.

PROPOSITION VII. THEOREM

**660.** *The plane passed through two diagonally opposite edges of a parallelepiped divides it into two equivalent triangular prisms.*



GIVEN—the parallelepiped  $ABCD-R$  divided by the plane  $ARTC$  into two triangular prisms  $ABC-S$  and  $ACD-U$ .

TO PROVE these triangular prisms are equivalent.

Let  $FGKL$  be a right section of the parallelepiped, cutting the plane  $ARTC$  in  $FK$ .

The planes  $AU$  and  $BT$  are parallel. § 551

Therefore  $FL$  and  $GK$  are parallel. § 544

Similarly  $FG$  and  $LK$  are parallel.

Therefore  $FGKL$  is a parallelogram. § 114

Hence the triangles  $FGK$  and  $FKL$  are equal. § 116

Now the triangular prism  $ABC-R$  is equivalent to a right prism whose base is the right section  $FGK$  and whose altitude is the lateral edge  $AR$ , and  $ACD-U$  is equivalent to a right prism whose base is  $FKL$  and whose altitude is  $AR$ .

§ 658

These two right prisms are equal. § 655

Therefore  $ABC-R$  and  $ACD-U$  are equivalent. Ax. I

Q. E. D.

## PROPOSITION VIII. THEOREM

**661.** *Two rectangular parallelepipeds having equal bases are to each other as their altitudes.*



**GIVEN**—the rectangular parallelepipeds  $P$  and  $P'$  having equal bases, their altitudes being  $AC$  and  $A'C'$ .

**TO PROVE**

$$\frac{P'}{P} = \frac{A'C'}{AC}.$$

**CASE I.** *When the altitudes are commensurable.*

Suppose the common measure of  $AC$  and  $A'C'$  to be contained in  $AC$  5 times, and in  $A'C'$  3 times.

Then 
$$\frac{A'C'}{AC} = \frac{3}{5}.$$

Through the points of division of  $AC$  and  $A'C'$  pass planes parallel to the bases.

These planes divide the parallelepipeds into smaller parallelepipeds, all of which are equal. §§ 631, 655

$P$  contains 5 and  $P'$  contains 3 of these small parallelo-  
pipeds.

Hence 
$$\frac{P'}{P} = \frac{3}{5}.$$

Therefore 
$$\frac{P'}{P} = \frac{A'C'}{AC}. \quad \text{Ax. 1}$$

CASE II. *When the altitudes are incommensurable.*



Divide  $AC$  into any number of equal parts and apply one of these parts to  $A'C'$  as often as  $A'C'$  will contain it.

Since  $AC$  and  $A'C'$  are incommensurable, there will be a remainder  $DC'$  less than one of these parts.

Pass a plane through  $D$  parallel to the bases of  $P'$  and let  $X$  be the rectangular parallelopiped between this plane and the lower base of  $P'$ .

Then, since  $A'D$  and  $AC$  are commensurable, we have

$$\frac{X}{P} = \frac{A'D}{AC}. \quad \text{Case I}$$



If each of the parts of  $AC$  be continually bisected, each part can be made as small as we please.

Therefore  $DC'$ , which is always less than one of these parts, can be made as small as we please.

But it can never be reduced to zero, since  $AC$  and  $A'C'$  are given incommensurable.

Therefore  $A'D$  will approach  $A'C'$  as a limit. § 185

Hence  $\frac{A'D}{AC}$  will approach  $\frac{A'C'}{AC}$  as a limit. § 190

Likewise  $\frac{X}{P}$  will approach  $\frac{P'}{P}$  as a limit.

Therefore  $\frac{P'}{P} = \frac{A'C'}{AC}$ . § 186

Q. E. D.

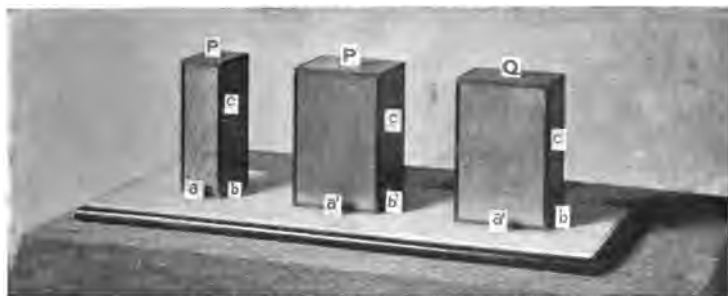
**662. Def.**—The three edges of a rectangular parallelepiped meeting at a common vertex are called its **dimensions**.

**663. Remark.**—The preceding theorem may be stated thus:

*Two rectangular parallelepipeds which have two dimensions in common are to each other as their third dimensions.*

PROPOSITION IX. THEOREM

**664.** *Two rectangular parallelepipeds which have one dimension in common are to each other as the products of the two other dimensions.*



**GIVEN**—the rectangular parallelepipeds  $P$  and  $P'$ , having the dimension  $c$  common, the other dimensions being  $a, b$  and  $a', b'$  respectively.

**TO PROVE** 
$$\frac{P}{P'} = \frac{a \times b}{a' \times b'}$$

Let  $Q$  be a third rectangular parallelepiped having the dimensions  $a', b, c$ .

Then  $P$  and  $Q$  have two dimensions  $b$  and  $c$  in common.

Hence 
$$\frac{P}{Q} = \frac{a}{a'}$$
 § 663

Also  $Q$  and  $P'$  have two dimensions  $a'$  and  $c$  in common.

Hence 
$$\frac{Q}{P'} = \frac{b}{b'}$$

Multiplying these equations together, we get

$$\frac{P}{P'} = \frac{a \times b}{a' \times b'}$$

Q. E. D.

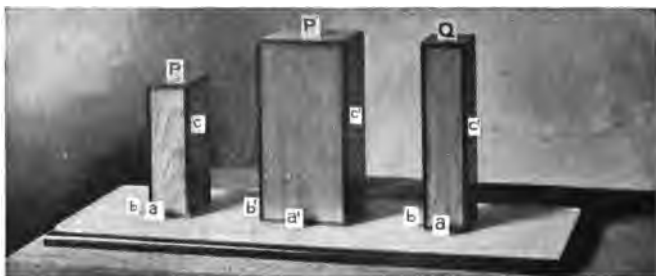


**665. Remark.**—This theorem may be stated thus:

*Two rectangular parallelepipeds having equal altitudes are to each other as their bases.*

PROPOSITION X. THEOREM

**666.** *Any two rectangular parallelepipeds are to each other as the products of their three dimensions.*



GIVEN—the rectangular parallelepipeds  $P$  and  $P'$ , whose dimensions are  $a, b, c$  and  $a', b', c'$  respectively.

TO PROVE 
$$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}$$

Let  $Q$  be a third rectangular parallelepiped having the dimensions  $a, b, c'$ .

Then 
$$\frac{P}{Q} = \frac{c}{c'} \quad \cdot \S 663$$

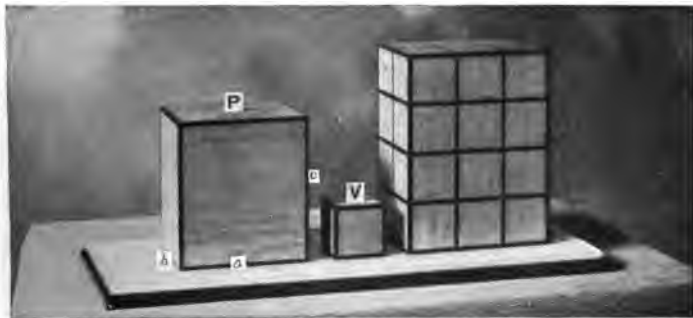
And 
$$\frac{Q}{P'} = \frac{a \times b}{a' \times b'} \quad \S 664$$

Multiplying these equations together, we have

$$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'} \quad \text{Q. E. D.}$$

PROPOSITION XI. THEOREM

**667.** *The volume of a rectangular parallelopiped is equal to the product of its three dimensions, provided that the unit of volume is a cube whose edge is the linear unit.*



*Proof.*—Let  $P$  be any rectangular parallelopiped whose dimensions are  $a$ ,  $b$ , and  $c$ , and let the cube  $V$ , whose edge is the linear unit, be the unit of volume.

Then  $\frac{P}{V}$  is the volume of  $P$ . § 656

But  $\frac{P}{V} = \frac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c$ . § 666

Therefore vol.  $P = a \times b \times c$ . Q. E. D.

**668. Remark.**—Hereafter the unit of volume is to be understood to be a cube whose edge is the linear unit.

**669. Remark.**—This theorem may also be stated :

*The volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

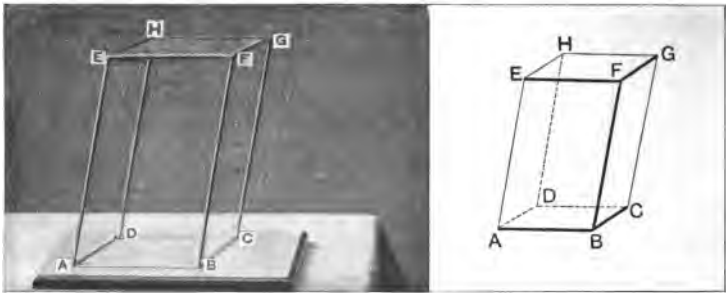
**670. COR. I.** *The volume of a cube is equal to the third power of its edge.*

Hence it is that the *third power* of a number is called the *cube* of that number.

**671. Remark.**—When the three dimensions of a rectangular parallelepiped are exactly divisible by the linear unit, the truth of the proposition may be rendered evident by dividing the parallelepiped into cubes, whose edges are equal to the linear unit.

Thus, if three edges which meet at a common vertex are respectively 2 units, 3 units, and 4 units in length, the parallelepiped may be divided into 24 cubes, each equal to the unit of volume, by passing planes perpendicular to the edges through their points of division.

**672. CONSTRUCTION.** *To construct a parallelepiped having as edges three given straight lines drawn from the same point.*



**GIVEN** the straight lines  $AB$ ,  $AD$ , and  $AE$ .

**TO CONSTRUCT**—a parallelepiped having them as edges.

Pass a plane through each pair of the given straight lines.

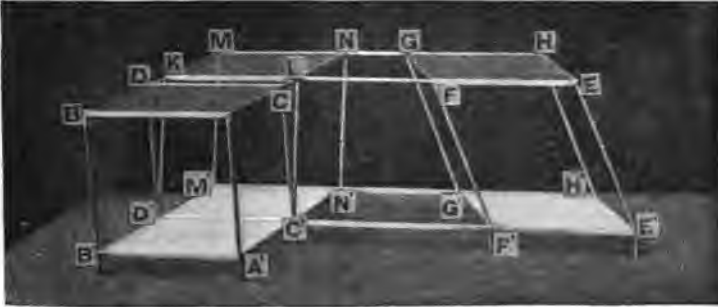
Then pass a plane through the extremity of each line parallel to the plane of the other two.

The solid thus formed may be shown to be a parallelepiped by applying successively §§ 544, 630, 633, 639.

**673. Exercise.**—Show that if the three given lines in the preceding construction are perpendicular to each other, the parallelepiped formed will be rectangular.

PROPOSITION XII. THEOREM

**674.** *The volume of any parallelepiped is equal to the product of its base and altitude.*



GIVEN—the *oblique* parallelepiped  $FH'$ , whose base is  $F'E'H'G'$  and altitude  $h$ .

TO PROVE  $\text{vol. } FH' = F'E'H'G' \times h$ .

Produce the edges  $EF, HG, E'F', H'G'$ , and in  $E'F'$  produced take  $C'D' = E'F'$ .

Through  $C'$  and  $D'$  pass planes perpendicular to  $E'D'$ , forming the *right* parallelepiped  $KN'$ .

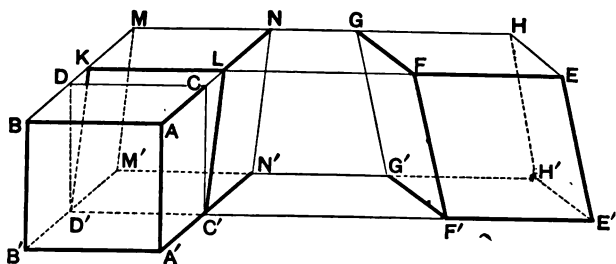
Now produce the edges  $N'C', NL, MK, M'D'$  of the right parallelepiped and in  $N'C'$  produced take  $C'A' = N'C'$ .

In the plane  $A'N$  draw  $C'C$  perpendicular to  $A'C'$ .

The three lines  $C'D', C'A', C'C$  are perpendicular to each other. § 530

Therefore the parallelepiped  $BC'$  formed upon them as edges will be rectangular. § 673

The rectangular and oblique parallelepipeds are each equivalent to the right parallelepiped, and therefore to each other. § 658



Their bases  $F'E'H'G'$  and  $B'A'C'D'$  are each equivalent to  $D'C'N'M'$ , and therefore to each other. § 386

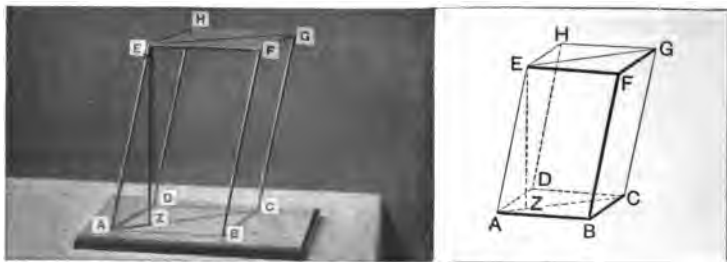
And the altitude of each is  $h$ . § 565

But  $\text{vol. } BC' = B'A'C'D' \times h$ . § 667

Therefore  $\text{vol. } FH' = F'E'H'G' \times h$ . Q. E. D.

### PROPOSITION XIII. THEOREM

**675.** *The volume of a triangular prism is equal to the product of its base and altitude.*



**GIVEN**—the triangular prism  $ABC-F$  having the base  $ABC$  and altitude  $EZ$ .

**TO PROVE**  $\text{vol. } ABC-F = ABC \times EZ$ .

Construct the parallelepiped  $ABCD-F$  having  $BA$ ,  $BC$ , and  $BF$  as edges. § 672

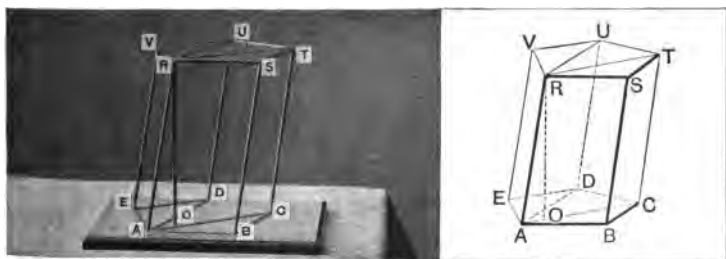
Then the volume of the paralleliped is the product of its base  $ABCD$  and its altitude  $EZ$ . § 674

But the volume of the triangular prism is half the volume of the paralleliped; its base is half the base of the paralleliped; and its altitude is the same. §§ 660, 116

Therefore the volume of the triangular prism is the product of its base  $ABC$  and its altitude  $EZ$ . Q. E. D.

PROPOSITION XIV. THEOREM

**676.** *The volume of any prism is equal to the product of its base and altitude.*



GIVEN—the prism  $ABCDE-R$  with base  $ABCDE$  and altitude  $RO$ .

TO PROVE  $\text{vol. } ABCDE-R = ABCDE \times RO$ .

The prism may be divided into triangular prisms by planes passed through  $AR$  and the diagonally opposite edges.

The volume of each triangular prism is the product of its base and altitude. § 675

They have the common altitude  $RO$ . § 565

Therefore the volume of the whole prism is the sum of the bases of the triangular prisms, i. e., the base of the whole prism, multiplied by the common altitude. Q. E. D.

**677. COR. I.** *Two prisms having equivalent bases and equal altitudes are equivalent.*

**678. COR. II.** *Any two prisms are to each other as the products of their bases and altitudes.*

*Hint.*—Prove as in § 387.

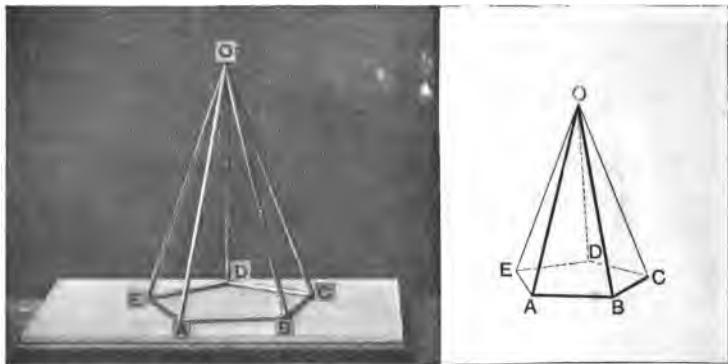
**679. COR. III.** *Two prisms having equivalent bases are to each other as their altitudes.*

**680. COR. IV.** *Two prisms having equal altitudes are to each other as their bases.*

### PYRAMIDS

**681. Defs.**—A **pyramid** is a polyhedron one of whose faces is a polygon and whose other faces are triangles meeting at the sides of the polygon for bases and a common vertex outside the plane of the polygon.

The polygon is the **base**; the triangles are the **lateral faces**; the common vertex of the triangles is the **vertex** of the pyramid; and the edges passing through the vertex are its **lateral edges**.



Thus  $ABCDE$  is the base;  $O$  is the vertex;  $OA, OB$ , etc., are the lateral edges; and  $OAB, OBC$ , etc., are the lateral faces of the pyramid  $O-ABCDE$ .

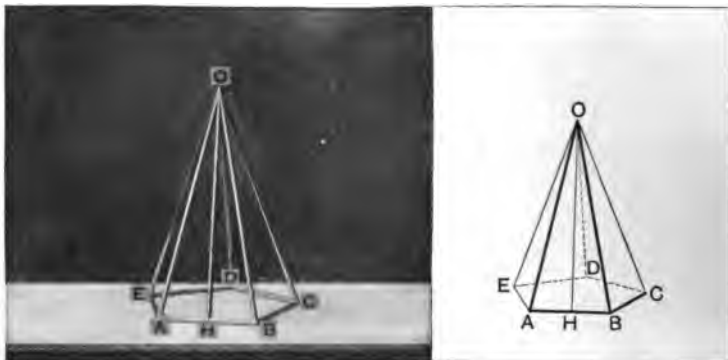




**687.** *Def.*—The lateral area of a pyramid is the sum of the areas of its lateral faces.

PROPOSITION XVI. THEOREM

**688.** *The lateral area of a regular pyramid is equal to one-half the product of the perimeter of its base and its slant height.*



GIVEN—the regular pyramid  $O-ABCDE$ , of which  $OH$  is the slant height.

TO PROVE—lat. area  $O-ABCDE = \frac{1}{2} (AB + BC + \text{etc.}) \times OH$ .

The lateral area of the pyramid is composed of the areas of the triangles  $OAB$ ,  $OBC$ , etc. § 687

The area of each triangle is half the product of its base and altitude.

Hence  $\text{area } OAB = \frac{1}{2} AB \times OH$ ,  
 $\text{area } OBC = \frac{1}{2} BC \times OH$ , etc. § 685

Therefore the lateral area of the pyramid is

$$\frac{1}{2} (AB + BC + \text{etc.}) \times OH. \text{ Q. E. D.}$$

**689.** *Def.*—A truncated pyramid is the portion of a pyramid contained between its base and a plane cutting all its lateral edges.

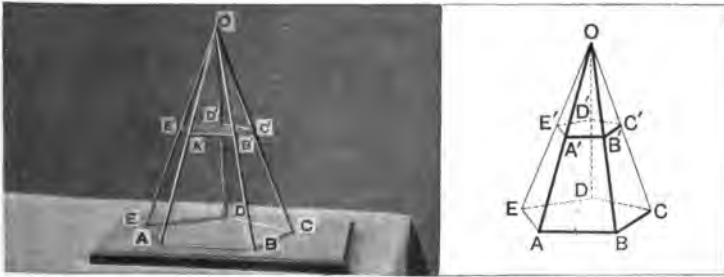
The section thus made, together with the base of the pyramid, are called the **bases** of the truncated pyramid.

The other faces are the **lateral faces** of the truncated pyramid.

**690. Def.**—A **frustum** of a pyramid is a truncated pyramid, the planes of whose bases are parallel.

PROPOSITION XVII. THEOREM

**691.** *The lateral faces of a frustum of a regular pyramid are equal trapezoids.*



**GIVEN**—the frustum  $EC'$  of the regular pyramid  $O-ABCDE$ .

**TO PROVE**—its faces are equal trapezoids, viz.:  $ABB'A' = BCC'B'$ , etc.

The faces *are* trapezoids, since  $A'B'$ ,  $B'C'$ , etc., are parallel to  $AB$ ,  $BC$ , etc., respectively. § 544

Superpose the equal isosceles triangles  $OAB$ ,  $OBC$  by turning the first over  $OB$  on to the second.

Then also must  $A'B'$  coincide with  $B'C'$ , both being parallel to  $BC$ . Ax. *b*

Thus the two trapezoids coincide and are equal. Likewise all the trapezoids are equal. Q. E. D.

**692. Def.**—The **slant height** of a frustum of a regular pyramid is the altitude of any lateral face.

**693. COR.** *The lateral area of a frustum of a regular pyramid equals one-half the product of the sum of the perimeters of its bases and its slant height.*

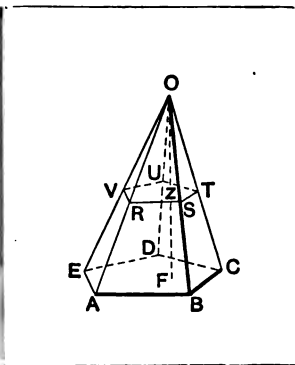
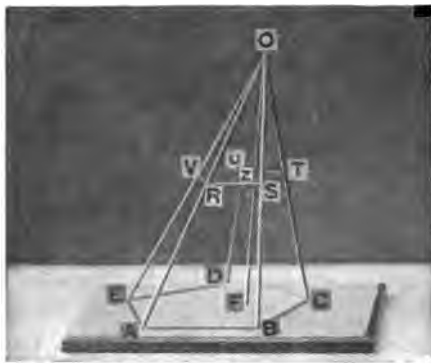
*Hint.*—Apply § 396.

**694. Def.**—The **altitude** of a pyramid is the perpendicular distance from the vertex to the plane of the base.

PROPOSITION XVIII. THEOREM

**695.** *If a pyramid is cut by a plane parallel to its base:*

- I. *The lateral edges and the altitude are divided proportionally.*
- II. *The section is a polygon similar to the base.*



GIVEN—the pyramid  $O-ABCDE$  cut by a plane parallel to its base  $ABCDE$  in the section  $RSTUV$ , and the altitude  $OF$  cutting the plane of the section in  $Z$ .

I. TO PROVE  $\frac{OR}{OA} = \frac{OS}{OB} = \frac{OT}{OC} = \text{etc.} = \frac{OZ}{OF}$ .

This follows immediately from § 556.

II. TO PROVE  $RSTUV$  is similar to  $ABCDE$ .

The corresponding sides of the two polygons are parallel.

§ 544

Hence their angles are equal.

§ 557

Also the triangles  $ORS$ ,  $OST$ , etc., are similar to  $OAB$ ,  $OBC$ , etc.

§ 275

Hence  $\frac{RS}{AB} = \frac{OS}{OB} = \frac{ST}{BC} = \text{etc.}$

Hence  $\frac{RS}{AB} = \frac{ST}{BC} = \text{etc.}$

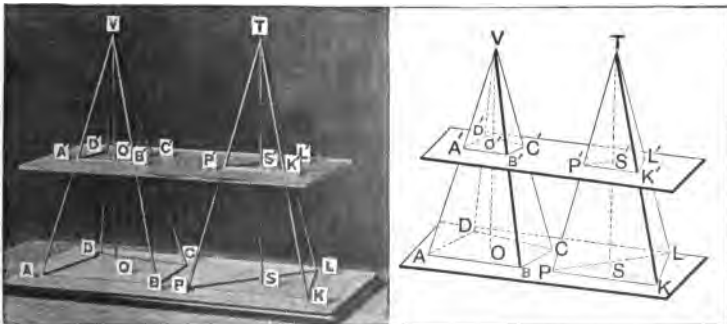
Therefore  $RSTUV$  is similar to  $ABCDE$ .

Q. E. D.

**696.** COR. I. *The areas of any sections of a pyramid parallel to its base are proportional to the squares of their distances from the vertex.*

OUTLINE PROOF:  $\frac{\text{area } RSTUV}{\text{area } ABCDE} = \frac{RS^2}{AB^2} = \frac{OS^2}{OB^2} = \frac{OZ^2}{OF^2}$ .

**697.** COR. II. *If two pyramids  $V-ABCD$  and  $T-PKL$ , having equal altitudes  $VO$  and  $TS$ , are cut by planes parallel to their bases at equal distances  $VO'$  and  $TS'$  from their vertices, the sections  $A'B'C'D'$  and  $P'K'L'$  thus formed will be proportional to the bases.*



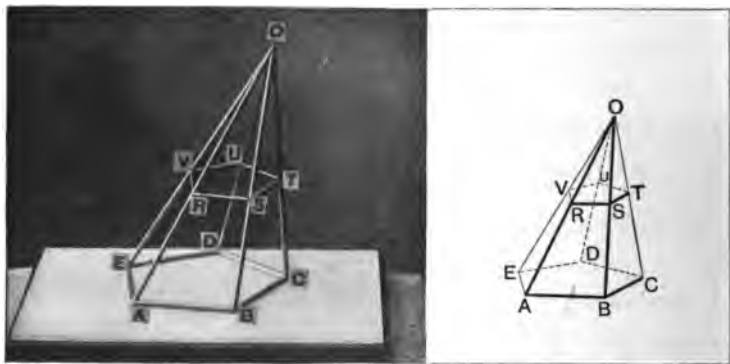
OUTLINE PROOF:  $\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{VO'^2}{VO^2} = \frac{TS'^2}{TS^2} = \frac{\text{area } P'K'L'}{\text{area } PKL}$ .

*Proof - see p. 313*

**698.** COR. III. *If two pyramids have equal altitudes and equivalent bases, sections parallel to their bases and equally distant from their vertices are equivalent.*

PROPOSITION XIX. THEOREM

**699.** *If the lateral edges of a pyramid are divided proportionally, the points of division lie in a plane parallel to the base of the pyramid.*



GIVEN—the pyramid  $O-ABCDE$  and the points  $R, S, T$ , etc., dividing the lateral edges so that  $\frac{OA}{OR} = \frac{OB}{OS} = \frac{OC}{OT} = \text{etc.}$

TO PROVE—that  $R, S, T$ , etc., lie in a plane parallel to the base.

Draw the straight lines  $RS, ST$ , etc.

In the triangle  $OAB$  the line  $RS$ , which divides the sides proportionally, is parallel to the base  $AB$ . § 273

Similarly  $ST$  is parallel to  $BC$ , etc.

Hence the plane of  $RS$  and  $ST$  is parallel to the base.

Similarly the plane of  $ST$  and  $TU$  is parallel to the base.

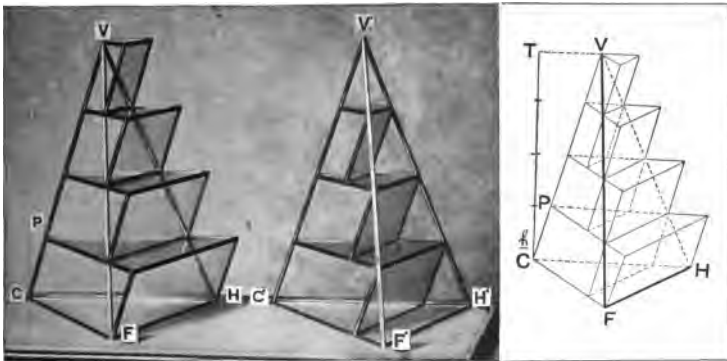
These planes coincide. § 554

In this way all the points  $R, S, T$ , etc., can be shown to lie in one plane parallel to the base. Q. E. D.

**700.** *Defs.*—A **triangular pyramid** is one whose base is a triangle; a **quadrangular pyramid**, one whose base is a quadrilateral.

PROPOSITION XX. THEOREM

**701.** *The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, when their number is indefinitely increased.*

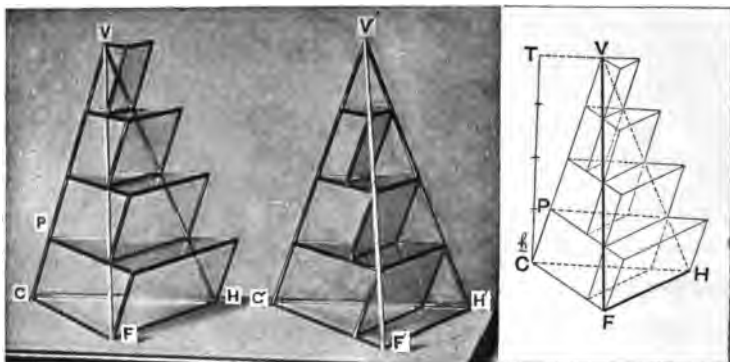


**GIVEN**—the triangular pyramid  $V-CFH$ , its altitude being  $CT$ .

**TO PROVE**—that its volume is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, when their number is indefinitely increased.

Divide the altitude  $CT$  into any number of equal parts and call one of these parts  $h$ .

Through the points of division pass planes parallel to the base, forming triangular sections. § 695 II



Upon the base  $CFH$  and upon the sections as *lower* bases construct prisms having their lateral edges parallel to  $VC$  and their altitudes equal to  $h$ .

This set of prisms may be said to be *circumscribed* about the pyramid.

Also with the sections as *upper* bases construct prisms having their lateral edges parallel to  $VC$  and their altitudes equal to  $h$ .

This set of prisms may be said to be *inscribed* in the pyramid.

The first circumscribed prism (beginning at the top) is equivalent to the first inscribed prism, the second circumscribed to the second inscribed, and so on until the last circumscribed remains.

§ 677

Hence the sum of the inscribed prisms differs from the sum of the circumscribed by the lower circumscribed prism  $P-CFH$ .

But the pyramid is intermediate between the total inscribed and the total circumscribed prisms.

Ax. 10

Therefore the difference between the pyramid and either

of these totals is less than the difference between the totals themselves, i. e., less than the lower circumscribed prism.

But the volume of this prism is the product of its base and altitude, and since its altitude can be indefinitely diminished, while its base remains the same, its volume can be made as small as we please. § 187

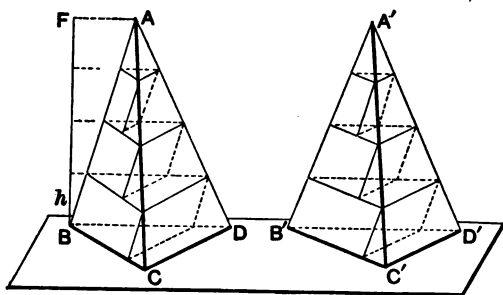
That is, the total of the inscribed prisms, or the total of the circumscribed prisms, can be made to differ from the pyramid by less than any assigned volume.

But they can never become equal to the pyramid. Ax. 10

Therefore the volume of the pyramid is their common limit. Q. E. D.

PROPOSITION XXI. THEOREM

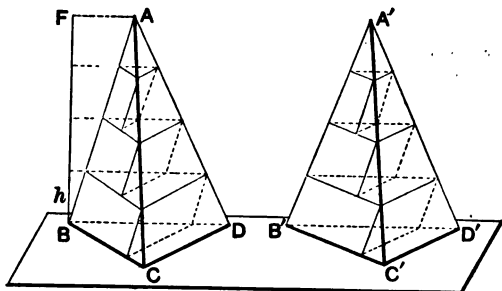
702. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



GIVEN—the triangular pyramids  $A-BCD$  and  $A'-B'C'D'$  having equivalent bases  $BCD$  and  $B'C'D'$  in the same plane and having a common altitude  $BF$ .

TO PROVE the pyramids are equivalent.





Divide  $BF$  into any number of equal parts and denote one of these parts by  $h$ .

Through the points of division pass planes parallel to the bases and cutting the two pyramids.

The corresponding sections made by these planes in the two pyramids will be equivalent. § 698

Inscribe in each pyramid a series of prisms having the sections as upper bases and having the common altitude  $h$ .

The corresponding prisms, having equal altitudes and equivalent bases, will be equivalent. § 677

Therefore the total volume (or  $S$ ) of the prisms inscribed in  $A-BCD$  will equal the total volume (or  $S'$ ) of the prisms inscribed in  $A'-B'C'D'$ .

Now suppose the number of divisions of the altitude  $BF$  to be indefinitely increased.

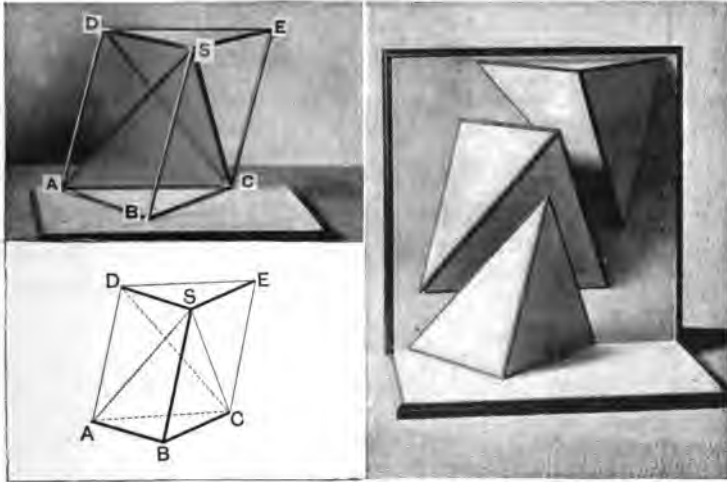
Then  $S$  will approach the volume of the pyramid  $A-BCD$  as a limit, and  $S'$  will approach the volume of the pyramid  $A'-B'C'D'$  as a limit. § 701

Since the variables  $S$  and  $S'$  are always equal to each other, their limits are equal. § 186

That is, the volumes of the pyramids are equal. Q. E. D.

PROPOSITION XXII. THEOREM

**703.** *The volume of a triangular pyramid is one-third the product of its base and altitude.*



**GIVEN** the triangular pyramid  $S-ABC$ .

**TO PROVE**—its volume is one-third its base  $ABC$  by its altitude.

Construct a triangular prism having  $ABC$  for its base and its lateral edges equal and parallel to  $BS$ .

Taking away the triangular pyramid  $S-ABC$  from the prism, we have left the quadrangular pyramid  $S-DACE$ .

Divide the latter by the plane  $SDC$  into two triangular pyramids  $S-DAC$  and  $S-DCE$ .

These pyramids have equal bases, the triangles  $DCA$  and  $DCE$ . § 116

They have equal altitudes, the perpendicular from the common vertex  $S$  upon the common plane of their bases.

Therefore they are equivalent.

§ 702

It can also be shown that the pyramids  $S-ABC$  and  $S-DAC$ , regarded as having the common vertex  $C$ , have equal bases and equal altitudes.

Hence these two pyramids are equivalent.

Hence all three are equivalent.

Therefore the pyramid  $S-ABC$  is one-third of the prism.

But the volume of the prism is the product of its base and altitude.

§ 675

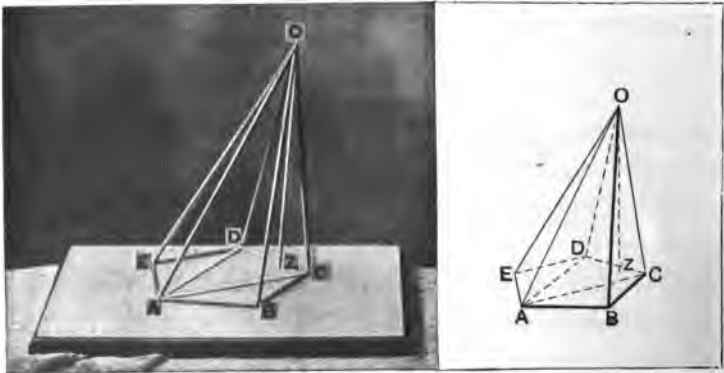
And the pyramid has the same base and altitude.

Hence the volume of the pyramid is one-third the product of its base and altitude.

Q. E. D.

PROPOSITION XXIII. THEOREM

**704.** *The volume of any pyramid is equal to one-third the product of its base and altitude.*



**GIVEN** the pyramid  $O-ABCDE$ , whose altitude is  $OZ$ .

**TO PROVE** vol.  $O-ABCDE = \frac{1}{3} ABCDE \times OZ$ .

Divide  $ABCDE$  into triangles by diagonals drawn from  $A$ .

Planes passed through  $OA$  and these diagonals will divide the pyramid into triangular pyramids;  $O-ABC$ ,  $O-ACD$ , and  $O-ADE$ .

The volume of each triangular pyramid is one-third the product of its base and the common altitude  $OZ$ . § 703

Therefore the volume of the whole pyramid is one-third the sum of the bases of the triangular pyramids, i. e., the base of the whole pyramid multiplied by the common altitude.

Q. E. D.

**705.** COR. I. *Pyramids having equivalent bases and equal altitudes are equivalent.*

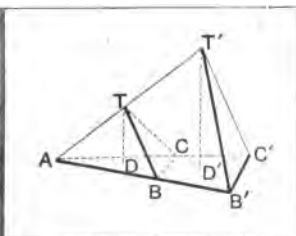
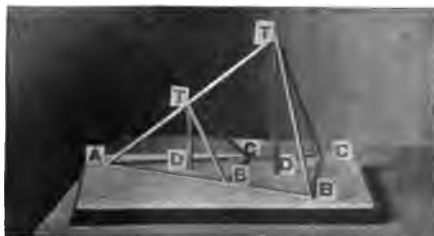
**706.** COR. II. *Any two pyramids are to each other as the products of their bases and altitudes.*

**707.** COR. III. *Two pyramids having equivalent bases are to each other as their altitudes.*

**708.** COR. IV. *Two pyramids having equal altitudes are to each other as their bases.*

PROPOSITION XXIV. THEOREM

**709.** *Two tetraedrons which have a triedral angle of one equal to a triedral angle of the other are to each other as the products of the three edges about the equal triedral angles.*





GIVEN—the tetrahedrons  $TABC$  and  $T'AB'C'$  having the triedral angle  $A$  in common. Let  $V$  and  $V'$  denote their respective volumes.

TO PROVE 
$$\frac{V}{V'} = \frac{AB \times AC \times AT}{AB' \times AC' \times AT'}.$$

From  $T$  and  $T'$  let fall the perpendiculars  $TD$  and  $T'D'$  upon the plane  $ABC$ .

The three points  $A$ ,  $D$ , and  $D'$  lie in one straight line.

§ 584

Now, considering  $ABC$  and  $AB'C'$  to be the bases of the tetrahedrons,

$$\frac{V}{V'} = \frac{ABC \times TD}{AB'C' \times T'D'} = \frac{ABC}{AB'C'} \times \frac{TD}{T'D'}. \quad \text{§ 706}$$

But 
$$\frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}. \quad \text{§ 398}$$

And since  $TD$  is parallel to  $T'D'$ , § 563  
the triangles  $ATD$  and  $AT'D'$  are similar. § 275

Hence 
$$\frac{TD}{T'D'} = \frac{AT}{AT'}.$$

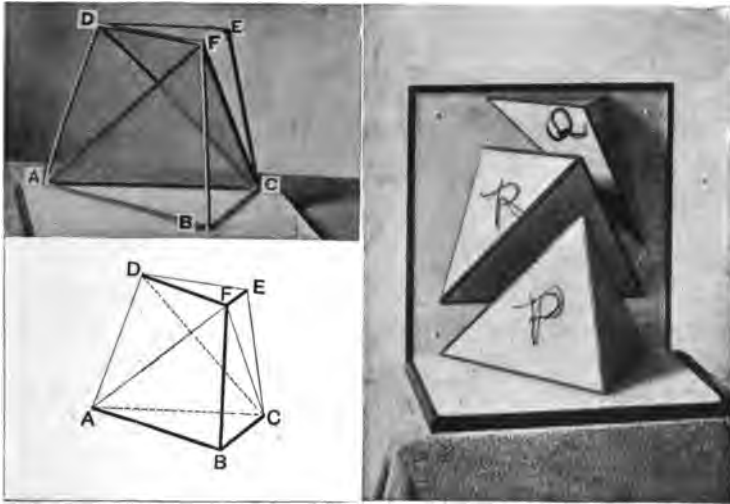
Therefore

$$\frac{V}{V'} = \frac{AB \times AC}{AB' \times AC'} \times \frac{AT}{AT'} = \frac{AB \times AC \times AT}{AB' \times AC' \times AT'}. \quad \text{Q. E. D.}$$

**710. Def.**—The **altitude** of a frustum of a pyramid is the perpendicular distance between the planes of its bases.

PROPOSITION XXV. THEOREM

**711.** *A frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



**GIVEN**—the frustum  $ABC-DEF$  of a triangular pyramid.

**TO PROVE**—it is equivalent to the sum of three pyramids, etc.

Pass a plane through  $F, A, C$ , and another through  $F, D, C$ , thus dividing the frustum into three triangular pyramids,  $F-ABC$ ,  $C-DEF$ , and  $F-DAC$ .

Call these pyramids  $P, Q$ , and  $R$  respectively, and represent  $ABC$  by  $B$ ,  $DEF$  by  $b$ , and the altitude of the frustum by  $h$ .

It is evident that if  $B$  and  $b$  be taken as the bases of  $P$  and  $Q$ , they have the common altitude  $h$ . § 565

Hence  $P = \frac{1}{3}h \times B$ , and  $Q = \frac{1}{3}h \times b$ . § 703

It remains to prove that  $R$  is equivalent to a pyramid whose altitude is  $h$  and whose base is  $\sqrt{B \times b}$ .

The pyramids  $P$  and  $R$ , regarded as having the common vertex  $C$  and their bases in the same plane, have the same altitude.

Hence 
$$\frac{P}{R} = \frac{ABF}{ADF}. \quad \text{§ 708}$$

But the triangles  $ABF$  and  $ADF$  have the same altitude, that of the trapezoid  $ABFD$ .

Hence 
$$\frac{ABF}{ADF} = \frac{AB}{DF}. \quad \text{§ 394}$$

Hence 
$$\frac{P}{R} = \frac{AB}{DF}. \quad \text{Ax. I}$$

Similarly 
$$\frac{R}{Q} = \frac{DAC}{DCE} = \frac{AC}{DE}.$$

Now the triangles  $ABC$  and  $DFE$  are similar. § 695 II

Hence 
$$\frac{AB}{DF} = \frac{AC}{DE}. \quad \text{§ 274}$$

Therefore 
$$\frac{P}{R} = \frac{R}{Q}. \quad \text{Ax. I}$$

Hence 
$$R^2 = P \times Q. \quad \text{§ 250}$$

Hence  $R = \sqrt{P \times Q} = \sqrt{\frac{1}{3}h \times B \times \frac{1}{3}h \times b} = \frac{1}{3}h \times \sqrt{B \times b}$ .

Therefore  $R$  is equivalent to a pyramid whose altitude is  $h$  and whose base is  $\sqrt{B \times b}$ . § 704

Q. E. D.

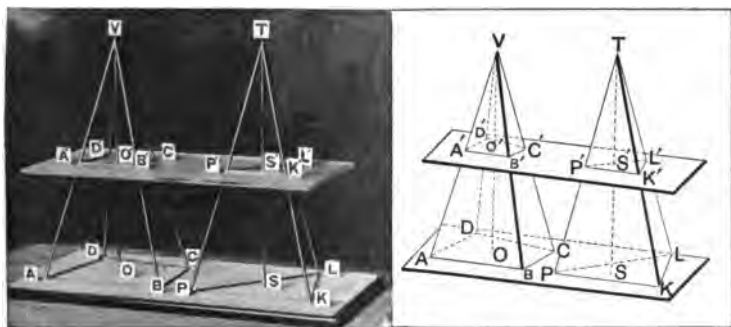
**712. Remark.**—If we denote the volume of the frustum by  $V$ , the proposition may be expressed in the form

$$V = \frac{1}{3}h(B + b + \sqrt{B \times b}).$$

*Question.*—Does it follow from Proposition XXV. that  $R$  is a pyramid whose altitude is  $h$  and base  $\sqrt{B \times b}$ ?

PROPOSITION XXVI. THEOREM

**713.** *A frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



**GIVEN** the frustum  $AC'$  of the pyramid  $V-ABCD$ .

Denote its lower and upper bases by  $B$  and  $b$  respectively, its altitude by  $h$ , and its volume by  $V$ .

**TO PROVE**  $V = \frac{1}{3}h(B + b + \sqrt{B \times b})$ .

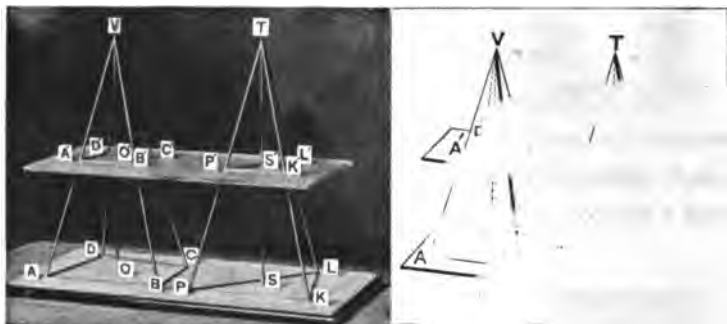
Let  $T-PKL$  be a triangular pyramid whose base is in the same plane as  $ABCD$  and equivalent to  $ABCD$ , whose vertex  $T$  is on the same side of this plane as  $V$  and whose altitude is equal to that of  $V-ABCD$ .

Prolong the plane of  $A'B'C'D'$  to cut  $T-PKL$  in the section  $P'K'L'$ .

Set  $B', b', h', V'$  for the lower base, upper base, altitude, and volume respectively of the triangular frustum  $PL'$ .

Then  $V' = \frac{1}{3}h'(B' + b' + \sqrt{B' \times b'})$ . (I) § 711





Now by hypothesis  $B' = B$ , and  $h' = h$

Moreover,  $b' = b$ .

Again  $V-ABCD$  and  $V-A'B'C'D'$  are respectively to  $T-PKL$  and  $T'-P'K'L'$ .

Taking away the small pyramids, the two frustums are equivalent.

Or  $V' = V$ .

Substituting for  $V'$ ,  $h'$ ,  $B'$ ,  $b'$ , their equal values in the formula,

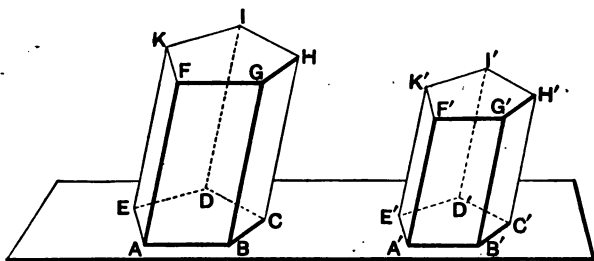
$$V = \frac{1}{3}h(B + b + \sqrt{Bb})$$

#### SIMILAR POLYEDRONS

**714. Def.**—Two polyhedrons are similar if they have the same number of faces similar each to its corresponding face, placed, and their homologous dihedrons similar.

#### PROPOSITION XXVI

**715.** *The ratio of any two homologous edges of similar polyhedrons is equal to the ratio of any other two homologous edges.*



GIVEN—the similar polyhedrons  $AH$  and  $A'H'$  in which any two edges  $AB$  and  $CH$  of one are respectively homologous to  $A'B'$  and  $C'H'$  of the other.

TO PROVE  $\frac{AB}{A'B'} = \frac{CH}{C'H'}$ .

Since the faces  $ABGF$  and  $A'B'G'F'$  are similar,

$$\frac{AB}{A'B'} = \frac{BG}{B'G'} \quad \S 274$$

Since the faces  $BCHG$  and  $B'C'H'G'$  are similar,

$$\frac{CH}{C'H'} = \frac{BG}{B'G'}$$

Therefore  $\frac{AB}{A'B'} = \frac{CH}{C'H'}$  Ax. I

Q. E. D.

**716. Def.**—The ratio of any two homologous edges of two similar polyhedrons is called the **ratio of similitude** of the polyhedrons.

**717. COR. I.** *The ratio of any two homologous faces of two similar polyhedrons is equal to the square of their ratio of similitude.*

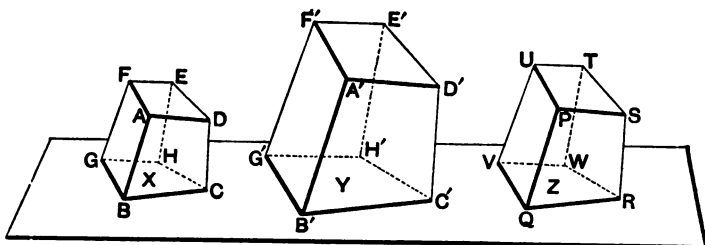
*Hint.*—Apply § 401.

**718. COR. II.** *The ratio of the total surfaces of two similar polyhedrons is equal to the square of their ratio of similitude.*

*Hint.*—Apply § 265.

## PROPOSITION XXVIII. THEOREM

**719.** *Two polyhedrons similar to a third are similar to each other.*



GIVEN—the polyhedrons  $AH$ , or  $X$ , and  $A'H'$ , or  $Y$ , both similar to  $PW$ , or  $Z$ .

TO PROVE that  $X$  is similar to  $Y$ .

The faces  $AC$  and  $A'C'$ , being both similar to  $PR$ , are similar to each other. § 294

In the same way all the faces of  $X$  may be shown similar to corresponding faces of  $Y$ .

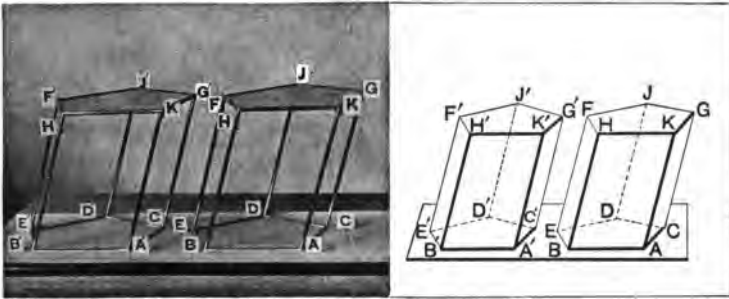
The polyhedrons  $X$  and  $Y$  also have the similar faces similarly arranged, since the arrangement in each is the same as the arrangement in  $Z$ .

Lastly, any two homologous dihedral angles of  $X$  and  $Y$ , being each equal to the same dihedral angle of  $Z$ , are equal to each other. Ax. 1

Therefore the polyhedrons  $X$  and  $Y$  are similar. § 714  
Q. E. D.

PROPOSITION XXIX. THEOREM

720. *Two similar polyedrons are equal, if their ratio of similitude is unity.*



GIVEN—the similar polyedrons  $AF$  and  $A'F'$  having

$$\frac{AB}{A'B'} = \frac{CG}{C'G'} = \text{etc.} = 1.$$

TO PROVE these polyedrons are equal.

The homologous faces are equal, being similar and having unity as a ratio of similitude. § 296

Superpose the faces  $ABHK$  and  $A'B'H'K'$ .

Since the dihedral angles  $AK$  and  $A'K'$  are equal, the planes of the faces  $CAKG$  and  $C'A'K'G'$  will coincide, and since the side  $AK$  of one already coincides with the side  $A'K'$  of the other, these faces, being equal, will coincide throughout.

In this way all the faces can be shown to coincide.

Therefore the polyedrons are equal.

Q. E. D.

## PROPOSITION XXX. THEOREM

**721.** *If two dihedral angles have their faces respectively parallel and extending in the same direction, they are equal.*

The proof is left to the student.

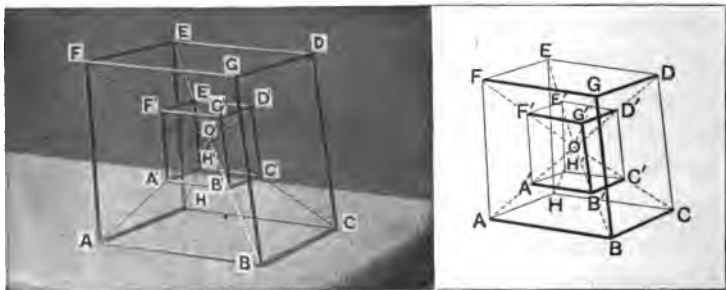
**722.** *Defs.*—If the vertices  $A, B, C, D$ , etc., of a polyedron are joined by straight lines to any point  $O$ , and the lines  $OA, OB, OC, OD$ , etc., are divided in the same ratio at the points  $A', B', C', D'$ , etc., the polyedron  $A'B'C'D'$ , etc., is said to be **radially situated** with regard to the polyedron  $ABCD$ , etc.

The ratio of the rays  $OA'$  and  $OA$  is called the **determining ratio**, or **ray ratio**, of the two polyedrons.

The point  $O$  is called the **ray centre**.

## PROPOSITION XXXI. THEOREM

**723.** *Two radially situated polyedrons are similar, and their ratio of similitude is equal to the ray ratio.*



**GIVEN**—the radially situated polyhedrons  $AD$  and  $A'D'$ ,  $O$  being the ray centre.

**TO PROVE**—that they are similar, and that the ray ratio is their ratio of similitude.

The two polyhedrons are made up of pyramids having  $O$  for common vertex.

In the pyramid  $O\text{-}ABCH$  the plane  $A'C'$  is parallel to the base. § 699

Hence the polygons  $A'C'$  and  $AC$  are similar. § 695 II

In the same way all the faces of one polyhedron can be shown to be similar to the corresponding faces of the other.

And the dihedral angles are equal, since their faces are respectively parallel and extending in the same direction from their edges. § 721

Therefore the polyhedrons are similar. § 714

Again, the triangles  $OA'B'$  and  $OAB$  are similar. § 285

Hence the ratio of similitude of the polyhedrons,  $\frac{A'B'}{AB}$ , is

equal to the ray ratio,  $\frac{OA'}{OA}$ .

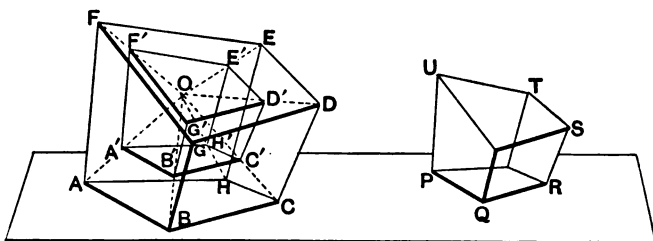
Q. E. D.

**724. Remark.**—The student should draw a figure with the ray centre outside of the polyhedrons and show that the proof is the same in this case. He should also draw a figure in which the ray centre is a common vertex of the two polyhedrons. The proof is slightly different with such a figure.

**725. Def.**—The ray centre is also called the **centre of similitude**.

## PROPOSITION XXXII. THEOREM

**726.** Any two similar polyhedrons can be radially placed, the ray ratio being equal to the ratio of similitude.



GIVEN the similar polyhedrons  $FC$  and  $UR$ .

TO PROVE—that they can be radially placed, the ray ratio being the ratio of similitude.

With any point  $O$  as ray centre form a polyhedron  $F'C'$  radially situated with regard to  $FC$ , having the ray ratio  $\frac{OA'}{OA}$  equal to the ratio of similitude  $\frac{PQ}{AB}$  of  $UR$  and  $FC$ .

Then  $F'C'$  and  $FC$  will be similar, the ratio of similitude  $\frac{A'B'}{AB}$  being equal to the ray ratio  $\frac{OA'}{OA}$ . § 723

But  $UR$  and  $FC$  are given similar, and their ratio of similitude is  $\frac{PQ}{AB}$ .

Therefore  $F'C'$  and  $UR$  are similar. § 719

Now since  $\frac{A'B'}{AB} = \frac{OA'}{OA}$ , and  $\frac{OA'}{OA} = \frac{PQ}{AB}$ ,

then  $\frac{A'B'}{AB} = \frac{PQ}{AB}$ .

By alternation  $\frac{A'B'}{PQ} = \frac{AB}{AB} = 1$ .

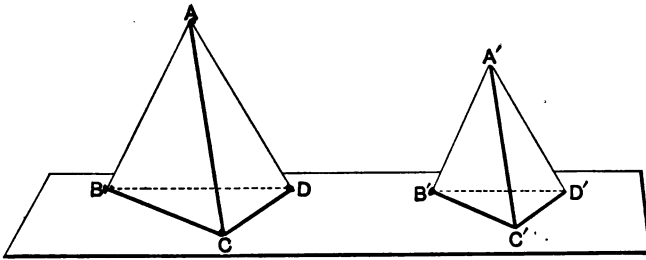
That is, the ratio of similitude of  $F'C'$  and  $UR$  is unity.

Therefore  $UR$  can be made to coincide with  $F'C'$ . § 720

In other words,  $FC$  and  $UR$  can be radially placed, the ray ratio being the ratio of similitude. Q. E. D.

PROPOSITION XXXIII. THEOREM

727. Two similar tetraedrons are to each other as the cubes of any two homologous edges.



GIVEN the similar tetraedrons  $ABCD$  and  $A'B'C'D'$ .

TO PROVE 
$$\frac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

The face angles of the triedral angles  $A$  and  $A'$  are equal each to each and similarly arranged. § 274

Hence these triedral angles are equal. § 597

Therefore 
$$\frac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \frac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'} \quad \text{§ 709}$$

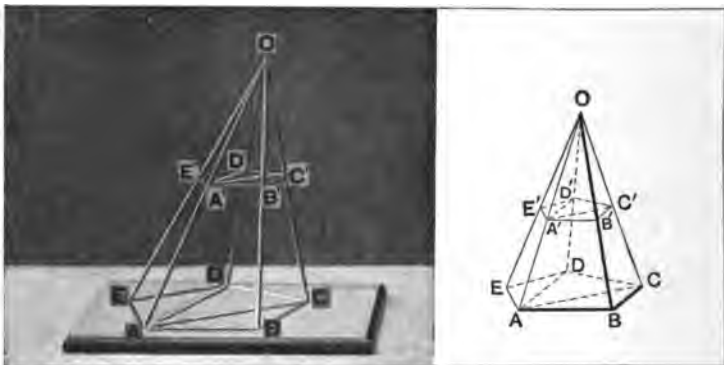
$$= \frac{AB}{A'B'} \times \frac{AC}{A'C'} \times \frac{AD}{A'D'}$$

$$= \frac{AB}{A'B'} \times \frac{AB}{A'B'} \times \frac{AB}{A'B'} \quad \text{§ 715}$$

That is, 
$$\frac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$
 Q. E. D.



**728. COR.** *If a pyramid is cut by a plane parallel to its base, the pyramid cut off is similar to the first, and the two pyramids are to each other as the cubes of any two homologous edges.*



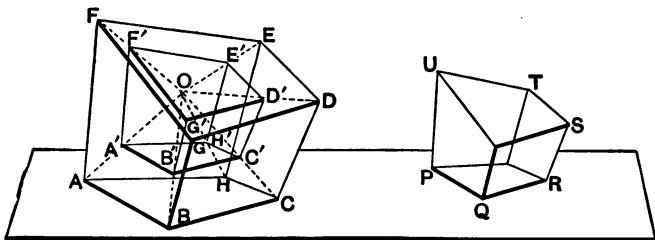
*Hint.*—Prove first that the lateral edges are divided proportionally. The pyramids are therefore similar (§ 723). Divide the pyramids into similar triangular pyramids as shown in the figure.

$$\text{Then } \frac{\overline{OA}^3}{\overline{OA'}^3} = \frac{\text{vol. } O-ABC}{\text{vol. } O-A'B'C'} = \frac{\text{vol. } O-ACD}{\text{vol. } O-A'C'D'} = \frac{\text{vol. } O-ADE}{\text{vol. } O-A'D'E'}$$

Now apply § 265.

PROPOSITION XXXIV. THEOREM

**729.** *The ratio of the volumes of any two similar polyedrons is equal to the cube of their ratio of similitude.*



GIVEN the two similar polyedrons  $FC$  and  $UR$ .

TO PROVE  $\frac{\text{vol. } FC}{\text{vol. } UR} = (\text{ratio of similitude})^3$ .

Suppose that  $UR$  is smaller than  $FC$ .

Place  $UR$  within  $FC$  in the position  $F'C'$ , radially situated with regard to  $FC$ , the ray centre being  $O$ . § 726

Then each polyedron can be divided into pyramids having  $O$  for common vertex and the faces of the polyedron for bases.

The planes  $A'B'C'H'$ ,  $B'C'D'G'$ , etc., are respectively parallel to the planes  $ABCH$ ,  $BCDG$ , etc. § 696

Hence

$$\frac{\text{vol. } O-ABCH}{\text{vol. } O-A'B'C'H'} = \frac{\overline{OA}^3}{\overline{OA'}^3} = \frac{\overline{OB}^3}{\overline{OB'}^3} = \frac{\text{vol. } O-BCDG}{\text{vol. } O-B'C'D'G'} = \text{etc.} \quad \text{§ 728}$$

Therefore

$$\frac{\text{vol. } O-ABCH + \text{vol. } O-BCDG + \text{etc.}}{\text{vol. } O-A'B'C'H' + \text{vol. } O-B'C'D'G' + \text{etc.}} = \frac{\overline{OA}^3}{\overline{OA'}^3}. \quad \text{§ 265}$$

That is,

$$\begin{aligned} \frac{\text{vol. } FC}{\text{vol. } UR} &= \frac{\overline{OA}^3}{\overline{OA'}^3} \\ &= (\text{ray ratio})^3 \\ &= (\text{ratio of similitude})^3. \quad \text{§ 726} \end{aligned}$$

Q. E. D.

REGULAR POLYEDRONS

**730.** *Def.*—A regular polyedron is one whose faces are equal regular polygons, and whose dihedral angles are all equal.

PROPOSITION XXXV. THEOREM

**731.** *Not more than five regular convex polyedrons are possible.*

*Proof.*—The faces of a regular polyedron must be regular polygons; at least three faces are necessary to form a polyedral angle; and the sum of the face angles of a convex polyedral angle must be less than  $360^\circ$ . Hence,

*Firstly*, since each angle of an *equilateral triangle* is equal to  $60^\circ$  (§ 64), either three, four, or five equilateral triangles can be combined to form a convex polyedral angle.

Not more than five equilateral triangles can be so combined, for six would make the sum of the face angles  $6 \times 60^\circ = 360^\circ$ , which is impossible.

Therefore not more than three regular convex polyedrons can be formed with triangular faces.

*Secondly*, since each angle of a *square* is equal to  $90^\circ$  (§ 114), three squares can be combined to form a convex polyedral angle.

Not more than three squares can be so combined, since  $4 \times 90^\circ = 360^\circ$ .

Therefore not more than one regular convex polyedron can be formed with square faces.

*Thirdly*, since each angle of a *regular pentagon* is equal to  $108^\circ$  (§ 463), three regular pentagons can be combined to form a convex polyedral angle.

Not more than three regular pentagons can be so combined, since  $4 \times 108^\circ = 432^\circ$ .

Therefore not more than one regular convex polyedron can be formed with pentagonal faces.

*Fourthly*, since each angle of a *regular hexagon* is equal to  $120^\circ$  (§ 463), no convex polyedral angle can be formed with regular hexagons, for  $3 \times 120^\circ = 360^\circ$ .

Similarly it can be shown that no convex polyedral angle can be formed with regular polygons of *more* than six sides.

Therefore not more than five regular convex polyhedrons can be formed. Q. E. D.

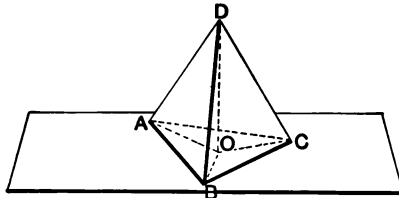
**732.** We will now show by actual construction that exactly five regular convex polyhedrons can be formed, viz. :

- (1.) The regular tetraedron, whose four faces are equilateral triangles.
- (2.) The regular hexaedron, or cube, whose six faces are squares.
- (3.) The regular octaedron, whose eight faces are equilateral triangles.
- (4.) The regular dodecaedron, whose twelve faces are regular pentagons.
- (5.) The regular icosaedron, whose twenty faces are equilateral triangles.



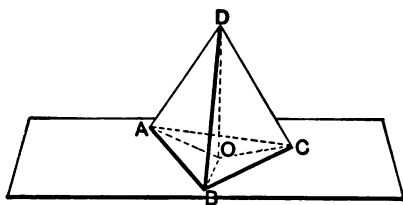
ICOSAEDRON      DODECAEDRON      OCTAEDRON      HEXAEDRON      TETRAEDRON

**733. CONSTRUCTION.** *To construct a regular tetraedron.*



Construct the equilateral triangle  $ABC$ .

At the centre  $O$  of the circumscribing circle erect the perpendicular  $OD$  to the plane  $ABC$ .



Take  $D$  so that  $AD=AB$ , and draw  $AD$ ,  $BD$ ,  $CD$ , and  $OA$ ,  $OB$ , and  $OC$ .

To PROVE that  $ABCD$  is a regular tetraedron.

Since  $O$  is the centre of  $ABC$ ,

$$OA = OB = OC.$$

Therefore  $AD = BD = CD$ . § 539 I

But  $AD$  was constructed equal to  $AB$ , and  $AB$  was given equal to  $BC$  and  $AC$ .

The four faces are therefore equal equilateral triangles.

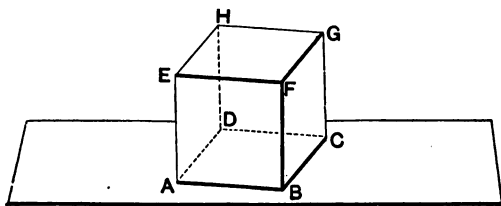
Also, since all the face angles of the four triedral angles are equal, all the triedral angles are equal. § 597

By superposing the triedral angles in pairs it may be seen that all the dihedral angles are equal.

Therefore  $ABCD$  is a regular tetraedron.

§ 730  
Q. E. F.

**734.** CONSTRUCTION. *To construct a regular hexaedron.*



Draw the three equal straight lines  $AB, AD,$  and  $AE$  perpendicular to each other.

Upon them construct a rectangular parallelepiped  $AG$ .

§ 673

The faces will all be squares.

§ 114

They will all be equal.

§ 377

That is, all the faces of the polyedron  $AG$  are equal regular quadrilaterals.

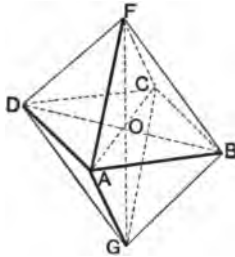
Its dihedral angles are all equal, since their plane angles are right angles.

§ 572

Therefore  $AG$  is a regular hexaedron.

Q. E. F.

**735.** CONSTRUCTION. *To construct a regular octaedron.*



Construct the square  $ABCD$ .

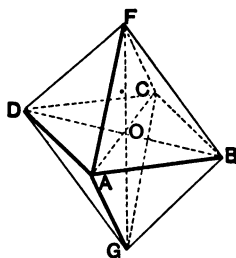
Through its centre  $O$  draw the straight line  $FG$  perpendicular to its plane.

In  $FG$  take two points  $F$  and  $G$  so that  $OF = OG = OB$ .

Join  $F$  and  $G$  to the points  $A, B, C, D$ .

**TO PROVE** that  $FABCDG$  is a regular octaedron.

The angles  $AOB$  and  $AOF$  are right angles and  $AO = OB = OF$ .



Therefore the triangles  $AOB$  and  $AOF$  are equal.

Hence  $AF=AB$ .

Also  $AF=BF=CF=DF=AG=BG=CG=DG$ .

The eight faces are therefore equal equilateral triangles.

Again, by construction  $FG$  and  $DB$  are equal and bisect each other at right angles.

Therefore  $DFBG$  is a square.

It is equal to  $ABCD$  and  $AO$  is perpendicular to its plane.

Hence the pyramids  $A-DFBG$  and  $F-ABCD$  are superposable and from symmetry each of the dihedral angles  $AD, AF, AB, AG$  can be made to coincide with each of the dihedral angles  $FA, FB, FC, FD$ .

Similarly any two dihedral angles can be shown to be equal.

Therefore  $FABCDG$  is a regular octahedron.

Q. E. F.

**736. CONSTRUCTION.** To construct a regular dodecaedron.

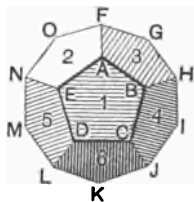


FIG. 1

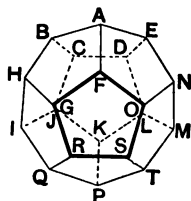


FIG. 2

Combine three equal regular pentagons  $ABCDE$ ,  $AENOF$ ,  $AFGHB$  so as to form a triedral angle at  $A$  (Fig. 1).

Pass a plane through  $H$ ,  $B$ , and  $C$ .

There will then be formed at  $B$  a triedral angle equal to that at  $A$ .

For, the diedral angle  $AB$  is common, and the face angles  $CBA$  and  $HBA$  of  $B$  are equal to the face angles  $FAB$  and  $EAB$  of  $A$ .

Hence the angle  $HBC$  is equal to an angle of a regular pentagon.

We can add, therefore, to the three pentagons already united a fourth regular pentagon,  $HBCJI$  having  $HBC$  for an angle.

Similarly we can add the fifth regular pentagon  $NEDLM$ .

Now the triedral angles at  $D$  and  $C$  can be shown to be equal to that at  $A$  by the process used above.

Hence the angles  $CDL$  and  $DCJ$  are each equal to an angle of a regular pentagon.

And  $LD$ ,  $DC$ , and  $CJ$  are in the same plane.

For the plane of  $LD$  and  $DC$  forms a diedral angle with face  $1$  at  $DC$  equal to that at  $AB$ , and the plane of  $DC$  and  $CJ$  forms a diedral angle with face  $1$  at  $DC$  equal to that at  $AB$ , and therefore these planes coincide.

We can therefore add to the five pentagons already joined a sixth regular pentagon  $LDCJK$  having  $LD$ ,  $DC$ , and  $CJ$  as sides.

Now, as we added the fourth pentagon, so we can add the seventh  $FOSRG$  (Fig. 2).

As we added the sixth, so we can add successively the eighth, ninth, and tenth  $OSTMN$ ,  $MTPKL$ ,  $KPQIJ$ .

Now, as we showed that the lines  $LD$ ,  $DC$ , and  $CJ$  were



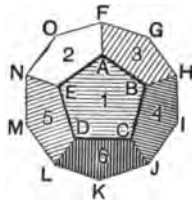


FIG. 1

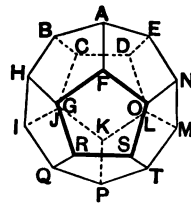


FIG. 2

in one plane, so we can show that the plane of  $GH$  and  $HI$  contains the lines  $GR$  and  $IQ$ .

The angles these lines make with each other can be shown as above equal to an angle of a regular pentagon.

We can therefore add the eleventh regular pentagon  $RGHIQ$ .

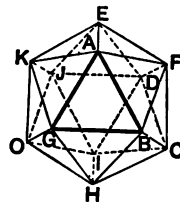
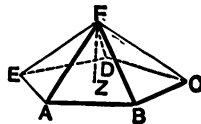
The twelfth pentagon can by the same methods be shown to be regular and equal to the others.

The dihedral angles are all easily seen to be equal.

Therefore the polyedron thus formed is a regular dodecaedron.

Q. E. F.

**737. CONSTRUCTION.** *To construct a regular icosaedron.*



Construct the regular pentagon  $ABCDE$ .

At its centre  $Z$  erect the perpendicular  $ZF$  to its plane, making  $AF=AB$ . Draw  $AF, BF, CF, DF, EF$ .

Then  $F-ABCDE$  is a regular pyramid and its five lateral faces are equal equilateral triangles.

Form nine other pyramids equal to  $F-ABCDE$ .

Now one of these can be made to coincide with  $F-ABCDE$  in five different ways. For it makes no difference which side of its base coincides with  $AB$ .

Hence all of the dihedral angles  $FA$ ,  $FB$ , etc., are equal to any one of the dihedral angles of the second pyramid, and are therefore all equal to each other.

Now place one of the seven pyramids, say  $A'-B'F'E'KG$ , so that the dihedral angle  $A'F'$  shall coincide with its equal  $AF$  and the faces  $A'F'B'$  and  $A'F'E'$  with their equals  $AFB$  and  $AFE$ ; thus adding the new faces  $EAK$ ,  $KAG$ , and  $GAB$ .

Place a second pyramid,  $B'-C'F'A'G'H$ , so that the dihedral angles  $B'F'$  and  $B'A'$  shall coincide with their equals  $BF$  and  $BA$ , and the faces  $B'C'F'$ ,  $B'F'A'$ , and  $B'A'G'$  with their equals  $BCF$ ,  $BFA$ , and  $BAG$ ; thus adding the new faces  $GBH$  and  $HBC$ .

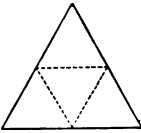
Similarly place two others,  $C'-D'F'B'H'I$  and  $D'-E'F'C'I'J$ , with their vertices at  $C$  and  $D$ ; thus adding the new faces  $HCI$ ,  $ICD$  and  $IDJ$ ,  $JDE$ .

Place a fifth,  $E'-K'A'F'D'J'$ , so that the dihedral angles  $E'A'$ ,  $E'F'$ , and  $E'D'$  shall coincide with their equals  $EA$ ,  $EF$ , and  $ED$ , and the faces  $E'A'K'$ ,  $E'F'A'$ ,  $E'D'F'$ , and  $E'J'D'$  with their equals  $EAK$ ,  $EFA$ ,  $EDF$ , and  $EJD$ ; thus adding the new face  $JEK$ .

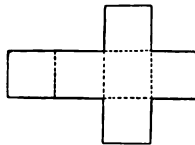
The four other pyramids can be similarly placed with their vertices at  $G$ ,  $H$ ,  $I$ , and  $J$ ; thus adding the new faces  $OGK$  and  $OGH$ ,  $OHI$ ,  $OIJ$ , and  $OJK$ .

The polyhedron thus completed, having twenty equal equilateral triangles for faces and having its dihedral angles all equal, is a regular icosaedron.

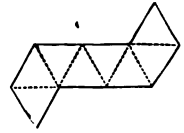
**738. Remark.**—The five regular polyedrons may be made from cardboard as follows: Draw on cardboard the figures given below, and on the inner lines cut the cardboard half through with a penknife. Cut the figures out entire and fold the cardboard as shown for the icosaedron in the accompanying plate.



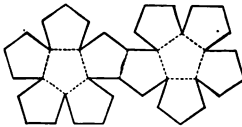
TETRAEDRON



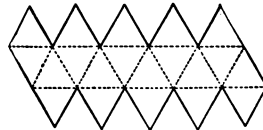
HEXAEDRON



OCTAEDRON



DODECAEDRON



ICOSAEDRON

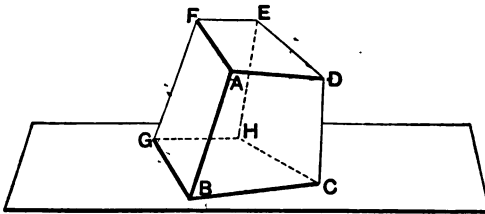


ICOSAEDRON

GENERAL THEOREMS ON POLYEDRONS

PROPOSITION XXXVI. THEOREM

**739.** *The number of the edges of any polyedron increased by two is equal to the number of its vertices increased by the number of its faces.\**



**GIVEN** any polyedron  $AH$ .

Denote the number of its edges by  $E$ ; the number of its vertices by  $V$ ; and the number of its faces by  $F$ .

**TO PROVE**  $E + 2 = V + F$ .

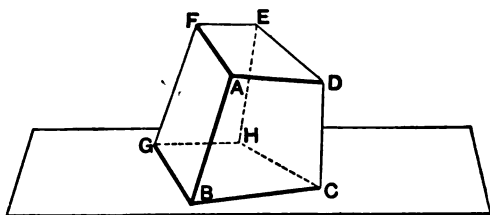
Let us put together the surface of the polyedron face by face and compare the number of edges with the number of vertices at each step.

If we take one face, as  $ABCD$ , the number of edges is obviously equal to the number of vertices.

That is, for one face,  $E = V$ .

Now let us add a second face, say a quadrilateral  $ABGF$ , to the first by placing the edges  $AB$  together. The new surface, consisting of  $ABCD$  and  $ABGF$ , will have three new edges,  $AF$ ,  $FG$ , and  $GB$ , and two new vertices,  $F$  and  $G$ .

\* This theorem was discovered by Euler (1707-1783).



The whole number of edges will be then one greater than the whole number of vertices.

However many sides the second face may have, it is easily seen that the number of new edges added will be one more than the number of new vertices.

Therefore for two faces,  $E = V + 1$ .

Next add a third face  $ADEF$  by placing an edge of it in coincidence with an edge of each of the first two faces.

We thus add two new edges,  $DE$  and  $FE$ , and one new vertex,  $E$ .

However many sides the third face may have, the increase in the number of edges is one more than the increase in the number of vertices.

Hence for three faces,  $E = V + 2$ .

We can in this way prove the following table:

For 1 face	$E = V$ .
For 2 faces	$E = V + 1$ .
For 3 faces	$E = V + 2$ .
For $m$ faces	$E = V + (m - 1)$ .
For $F - 1$ faces	$E = V + F - 2$ .

When the number of faces is  $F - 1$ , the surface is not closed.

To close it we add the last face.

In so doing we place each edge and each vertex of the

last face in coincidence with an edge and vertex of the open surface.

Adding the last face then increases neither the number of edges nor the number of vertices.

That is, for  $F$  faces,  $E = V + F - 2$ .

or  $E + 2 = V + F$ .

Q. E. D.

PROPOSITION XXXVII. THEOREM

**740.** *The sum of the angles of all the faces of any convex polyedron is equal to four right angles taken as many times as the polyedron has vertices less two.*

Let  $S$  denote the sum of the angles of all the faces, and  $V$  the number of vertices of any convex polyedron. Also let  $R$  denote a right angle.

TO PROVE  $S = 4R(V - 2)$ .

Any one face is a convex polygon.

Let the number of its sides be  $n$ .

Produce the sides in succession as in § 69.

The sum of the exterior angles thus formed is  $4R$ .

The sum of the interior and exterior angles is  $2R \times n$ . § 22

Do the same for all the faces of the polyedron considered as independent polygons of  $n, n', n''$ , etc., sides.

Then the sum of the exterior angles of the  $F$  faces is  $4R \times F$ .

The sum  $S$  of their interior angles plus the sum of their exterior angles is  $2R(n + n' + n'' + \text{etc.})$ .

That is,  $S + 4R \times F = 2R(n + n' + n'' + \text{etc.})$ .

Now, if  $E$  denotes the number of edges of the polyedron,

$$n + n' + n'' + \text{etc.} = 2E,$$

since each edge is a side of two polygons.

Hence  $S + 4R \times F = 2R \times 2E,$   
 or  $S = 4R(E - F).$

But by Euler's Theorem

$$E - F = V - 2.$$

Therefore

$$S = 4R(V - 2).$$

Q. E. D

#### PROBLEMS OF DEMONSTRATION

**741. Exercise.**—The four diagonals of a paralleloiped bisect each other.

**742. Exercise.**—Any straight line drawn through the intersection of the diagonals of a paralleloiped and terminated by two opposite faces is bisected in that point.

**743. Exercise.**—The sum of the squares of the four diagonals of a paralleloiped is equal to the sum of the squares of its twelve edges.

**744. Exercise.**—In a rectangular paralleloiped, the four diagonals are equal to each other; and the square of a diagonal is equal to the sum of the squares of the three edges which meet at a common vertex.

**745. Exercise.**—In a quadrangular prism two diagonals which connect two opposite lateral edges, bisect each other.

**746. Exercise.**—In any quadrangular prism the sum of the squares of the four diagonals plus eight times the square of the straight line joining the common middle points of the pairs of diagonals which bisect each other is equal to the sum of the squares of the twelve edges.

**747. Exercise.**—If a plane parallel to two opposite edges of a tetraedron cut the tetraedron, the section is a parallelogram.

**748. Exercise.**—If the angles at the vertex of a triangu-

lar pyramid are right angles, and the lateral edges are equal, prove that the sum of the perpendiculars on the lateral faces from any point in the base is constant.

**749. Exercise.**—The straight lines joining each vertex of a tetraedron with the intersection of the medians of the opposite face, meet in a point which divides each line into segments whose ratio is 3:1.

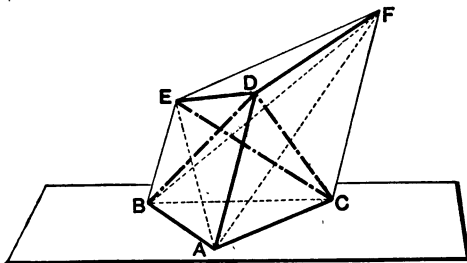
This point is called in Physics the *centre of gravity* of the tetraedron.

**750. Exercise.**—The straight lines joining the middle points of the opposite edges of a tetraedron meet in a point and are each bisected by the point.

**751. Exercise.**—A plane bisecting two opposite edges of a regular tetraedron divides the tetraedron into two equal polyedrons.

**752. Exercise.**—The pyramid whose base is one of the faces of a cube, and whose vertex is at the centre of the cube, is one-sixth part of the cube.

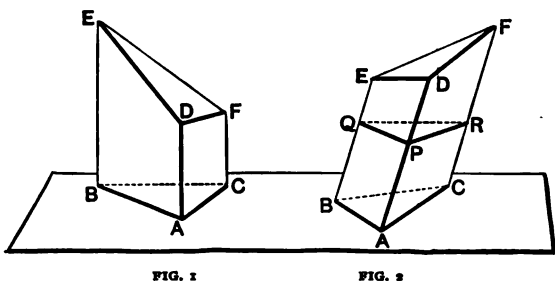
**753. Exercise.**—A truncated triangular prism is equivalent to the sum of three pyramids whose common base is either base of the truncated prism, and whose vertices are the three vertices of the other base.



*Hint.*—Divide the truncated triangular prism into three triangular pyramids by the planes  $DBC$  and  $DEC$ . Show that the pyramids  $D-BEC$  and  $E-ABC$  are equivalent. Also the pyramids  $D-CEF$  and  $F-ABC$ .



**754. Exercise.**—The volume of a right truncated triangular prism (Fig. 1) is equal to the product of one-third the sum of its lateral edges by the area of the base to which those edges are perpendicular.



**755. Exercise.**—The volume of any truncated triangular prism (Fig. 2) is equal to the product of one-third the sum of its lateral edges by the area of a right section.

**756. Exercise.**—The volume of a truncated prism, one of whose bases is a parallelogram, is equal to the product of a right section by one-fourth the sum of the lateral edges.

**757. Exercise.**—The volume of a truncated triangular prism is equal to the product of the lower base by the perpendicular on the lower base from the intersection of the medians of the upper base.

**758. Exercise.**—The perpendicular from a vertex of a regular tetraedron on the opposite face is three times the perpendicular from its own foot on any of the other faces.

#### PROBLEMS OF CONSTRUCTION

**759. Exercise.**—Pass a plane through a straight line given in position which shall divide a given parallelepiped into two equivalent polyhedrons.

**760. Exercise.**—Cut a cube by a plane so that the section shall be a regular hexagon.

**761. Exercise.**—Pass a plane through a given straight line which shall divide a given triangular prism into two equivalent truncated prisms.

**762. Exercise.**—Construct a parallelepiped of which three edges lie upon three given straight lines in space.

**763. Exercise.**—Pass a plane through a given point which shall divide a given regular tetraedron into two equal parts.

#### PROBLEMS FOR COMPUTATION

**764.** (1.) A rectangular block of marble is 1 m. 9 dcm. long, 9 dcm. 6 cm. broad, and 8 dcm. 9 cm. thick. What is its weight, if a cubic meter weighs 2675 kg.?

(2.) A barn with a gable roof is 60 ft. long, 30 ft. broad; the height from the floor to the eaves is 25 ft., to the gable  $32\frac{1}{2}$  ft. Find its contents.

(3.) The area of the base of a right prism is 12 sq. in., its total area is 295 sq. in.; the base is a regular hexagon. What is the volume?

(4.) The great pyramid is estimated to have cost ten dollars a cubic yard, and three dollars besides for each square yard of surface; in this estimate the lateral faces are considered to be planes. The altitude of the pyramid is 488 ft., its base is 764 ft. square. What was its cost?

(5.) Express the volume of a cube in terms of the length of a diagonal.

(6.) What is the ratio of an edge of a cube to that of a regular tetraedron of the same volume?

(7.) The area of the lower base of a frustum of a pyramid is 100 sq. cm., of the upper base 30 sq. cm., and the altitude of the frustum is 5 dcm. What would be the altitude of the complete pyramid?

(8.) What is the volume of a frustum of a regular triangular pyramid, if its slant height is 3.5 ft., a side of the lower base 4 ft., of the upper base 1.5 ft.?

(9.) The total surface of a regular tetraedron is 400 sq. ft. What is its volume?

(10.) The area of a face of a regular octaedron is 1 sq. ft. What is its volume?

(11.) What is the ratio of the lateral area of a regular tetraedron to the lateral area of a prism constructed upon the same base and having one of its lateral edges coincident with an edge of the tetraedron?

(12.) Find the volume of a truncated triangular prism, if the sides of a right section are respectively 2.416, 3.213, 1.963 in., and its lateral edges are 7.645, 6.633, 2.742 in.

# GEOMETRY OF SPACE

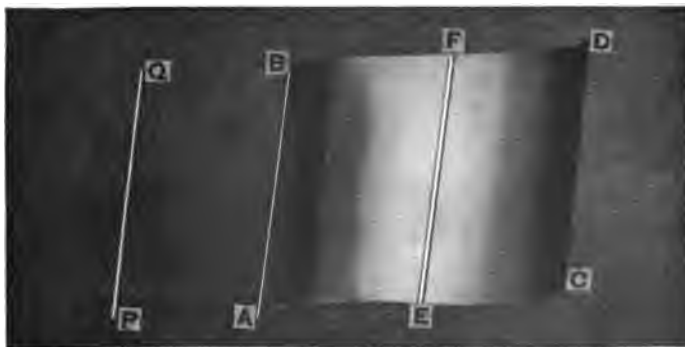
## BOOK VIII

### THE CYLINDER

**765. Def.**—A **curved line**, or **curve**, is a line no part of which is straight.

The curve may or may not lie entirely in one plane. An example of the first kind is the circumference of a circle; an example of the second kind is a curve like a corkscrew.

**766. Def.**—A **cylindrical surface** is a surface generated by a moving straight line which continually intersects a given fixed curve and is constantly parallel to a given fixed straight line.



Thus, if the straight line  $AB$  moves so as continually to intersect the curve  $AC$  and remains parallel to the line  $PQ$ , the surface generated,  $ABDC$ , is a cylindrical surface.

**767. Defs.**—The moving line is called the **generatrix**; the fixed curve is called the **directrix**.

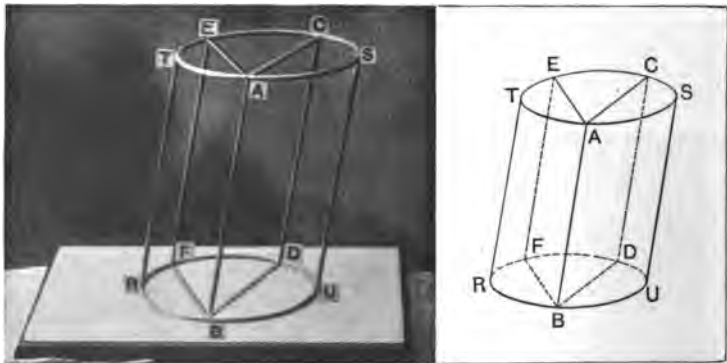
Any one position of the generatrix, as  $EF$ , is called an **element** of the surface.

**768. Remark.**—The generatrix is usually supposed to be indefinite in extent, so that the surface generated is also of indefinite extent.

The directrix may be any curve whatever. But for the student who has not studied the appendix the proofs are rigorous only when the directrix is considered to be the circumference of a circle.

PROPOSITION I. THEOREM

**769.** *The sections of a closed cylindrical surface made by two parallel planes cutting the elements are equal.*



**GIVEN**—the closed cylindrical surface  $RS$  cut by two parallel planes, not parallel to the elements, in the sections  $TS$  and  $RU$ .

**TO PROVE** that  $TS$  and  $RU$  are equal.

Let  $A$ ,  $C$ , and  $E$  be any three points in the perimeter of the upper section, and  $AB$ ,  $CD$ , and  $EF$  the corresponding

elements;  $B$ ,  $D$ , and  $F$  being the points where these elements meet the perimeter of the lower section.

Through  $AB$  and  $CD$  pass a plane. Pass another through  $AB$  and  $EF$ .

Then  $AC$  is parallel to  $BD$  and  $AE$  to  $BF$ . § 544

Hence  $AC=BD$  and  $AE=BF$ . § 118

The angles  $CAE$  and  $DBF$  are also equal. § 557

If, therefore, the planes of the two sections be superposed so that  $BD$  shall coincide with  $AC$ ,  $F$  will fall on  $E$ .

Now, if we suppose  $AC$  to be fixed and the point  $E$  to describe the perimeter of the upper section, then  $F$  will describe the perimeter of the lower section.

But in the superposed position of the sections  $F$  would always coincide with  $E$ .

Hence the perimeters of the two sections would coincide throughout. Therefore the sections are equal. Q. E. D.

**770. Defs.**—A **cylinder** is a solid bounded by a closed cylindrical surface and two parallel planes.

The cylindrical surface is called the **lateral surface**, and the equal sections formed by the parallel planes the **bases** of the cylinder.

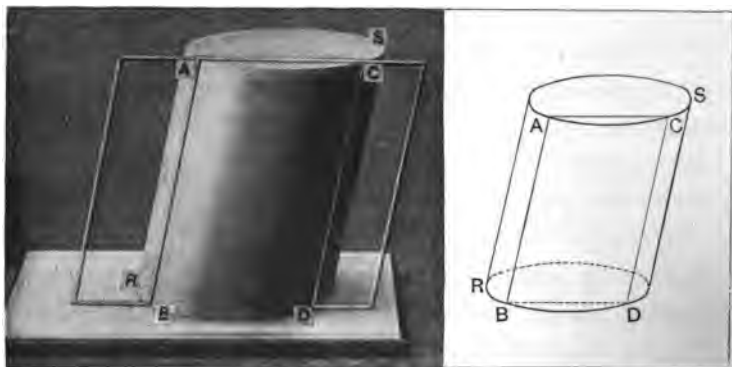


CYLINDERS

The term **element** of a cylinder is used to signify an element of its lateral surface.

## PROPOSITION II. THEOREM

**771.** *Every section of a cylinder made by a plane passing through an element is a parallelogram.*



GIVEN—the cylinder  $RS$  of which  $ABDC$  is a section made by a plane passing through an element  $AB$ .

TO PROVE  $ABDC$  is a parallelogram.

First, the lines  $AC$  and  $BD$  are straight and parallel.

§§ 528, 544

Since  $BA$  is an element, and therefore straight, we have only to prove that  $DC$  is straight and is parallel to  $BA$ .

Through  $D$  draw a straight line parallel to  $BA$ .

This line will lie in the cylindrical surface, by definition.

It will also lie in the plane determined by  $BA$  and  $D$ .

§ 526 II, IV

It therefore coincides with  $DC$ .

Hence  $DC$  is straight and is parallel to  $BA$ .

Therefore  $ABDC$  is a parallelogram.

§ 114  
Q. E. D.

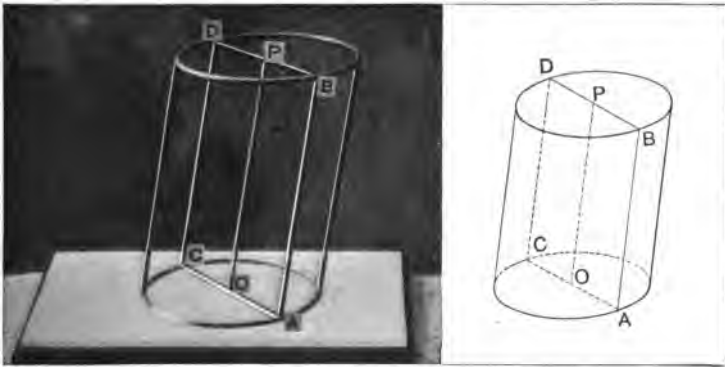
**772. Def.**—A **right cylinder** is one whose elements are perpendicular to its bases.

**773. COR.** *Every section of a right cylinder made by a plane perpendicular to its base is a rectangle.*

**774. Defs.**—A **circular cylinder** is one whose bases are circles. The straight line joining the centres of its bases is called the **axis** of the circular cylinder.

PROPOSITION III. THEOREM

**775.** *The axis of a circular cylinder is equal and parallel to its elements.*



**GIVEN** a circular cylinder  $AD$ , whose axis is  $OP$ .

**TO PROVE**— $OP$  is equal and parallel to any element  $AB$ .

Draw through  $B$  and  $P$  the diameter  $BD$  of the upper base, and let  $CD$  be the element passing through  $D$ .

Then pass a plane through  $AB$  and  $CD$  cutting the lower base in  $AC$ .

We have  $AC$  parallel to  $BD$ .

§ 544

Hence  $AC = BD$ .

§ 118



Therefore  $AC$  passes through  $O$  and is a diameter of the lower base. § 170

Hence  $AO = BP$ . § 158

Also  $AO$  was proved parallel to  $BP$ .

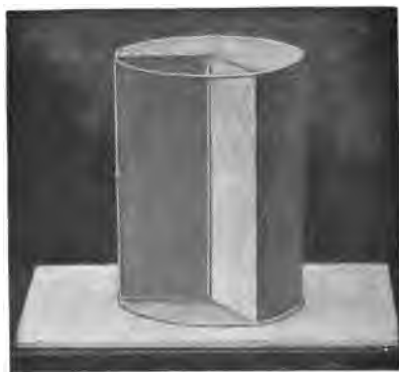
Hence the figure  $ABPO$  is a parallelogram. § 126

Therefore  $OP$  is equal and parallel to  $AB$ . § 117

Q. E. D.

**776. COR. I.** *The axis of a circular cylinder passes through the centres of all sections parallel to its base.*

**777. COR. II.** *A right circular cylinder may be generated by the revolution of a rectangle about one of its sides as an axis.*



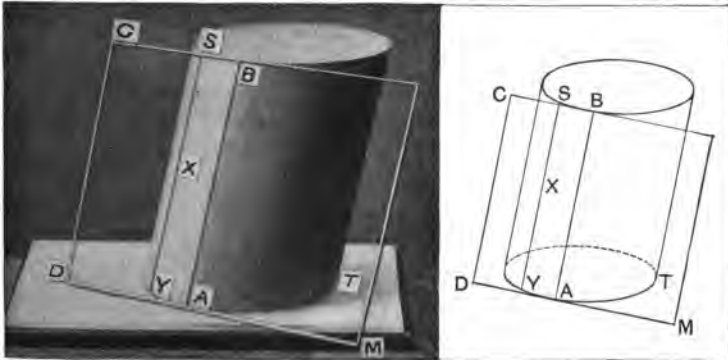
**778. Defs.**—For this reason a right circular cylinder is also called a **cylinder of revolution**.

The radius of the base of a cylinder of revolution is called the **radius of the cylinder**.

**779. Def.**—A plane is **tangent** to a cylinder when it passes through an element and meets its surface nowhere else.

PROPOSITION IV. THEOREM

**780.** *A plane passing through a tangent to the base of a cylinder and the element drawn at the point of contact is tangent to the cylinder.*



GIVEN—the cylinder  $ST$ , the tangent  $AD$  to its base, and the element  $AB$  drawn through the point of contact.

TO PROVE—that the plane  $CM$ , passing through  $AD$  and  $AB$ , is tangent to the cylinder.

If the plane should meet the surface of the cylinder in any point  $X$ , not in  $AB$ , draw the element  $SY$  passing through  $X$ .

Then  $SY$  would lie in the plane  $CM$ . § 526 II, IV

Therefore  $AD$  would meet the curve  $AT$  in two points,  $A$  and  $Y$ .

This cannot be, since  $AD$  is tangent to the base.

Hence the plane  $CM$  does not meet the surface of the cylinder except in  $AB$ .

It is therefore tangent to the cylinder.

§ 779  
Q. E. D.

**781.** COR. I. *Through a given element, one and only one plane tangent to the cylinder can be drawn.*

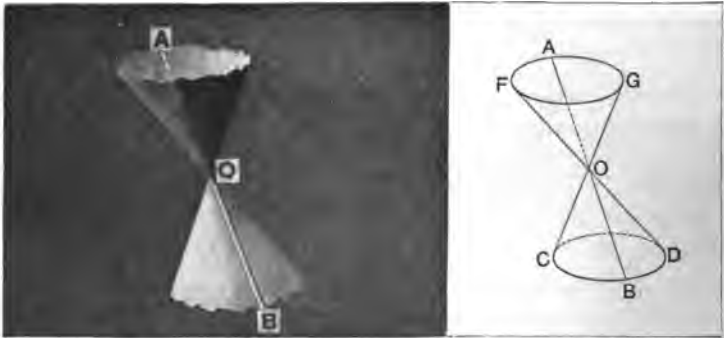
**782.** COR. II. *If a plane is tangent to a cylinder, its intersection with the plane of the base is tangent to the base.*

**783.** COR. III. *The intersection of two planes tangent to a cylinder is parallel to the elements.*

**784.** *Exercise.*—Show how to draw a plane through a given point tangent to a cylinder.

### THE CONE

**785.** *Def.*—A conical surface is a surface generated by a moving straight line which continually intersects a given fixed curve and constantly passes through a given fixed point.



Thus, if the straight line  $OB$  passes through the point  $O$  and moves so as continually to intersect the curve  $CD$ , the surface generated  $O-CBD$  is a conical surface.

**786.** *Def.*—The moving line is called the **generatrix**; the fixed curve the **directrix**; the fixed point the **vertex**.

Any straight line in the surface, as  $OA$ , representing one position of the generatrix, is called an **element** of the surface.

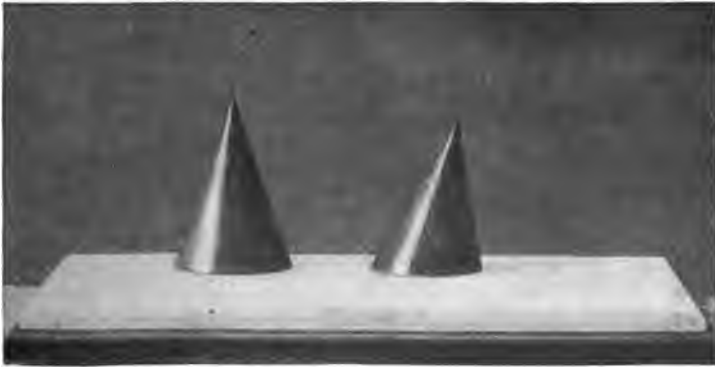
**787. Remark.**—If the generatrix is of indefinite length, as  $BOA$ , the conical surface consists of two symmetrical parts, each of indefinite extent, lying on opposite sides of the vertex, as  $O-CBD$  and  $O-GAF$ .

The directrix may be any curve whatever. But for the student who has not studied the appendix the proofs are rigorous only when the directrix is considered to be the circumference of a circle.

**788. Defs.**—A **cone** is a solid bounded by a closed conical surface and a plane.

The conical surface is called the **lateral surface** and the section made by the plane the **base** of the cone.

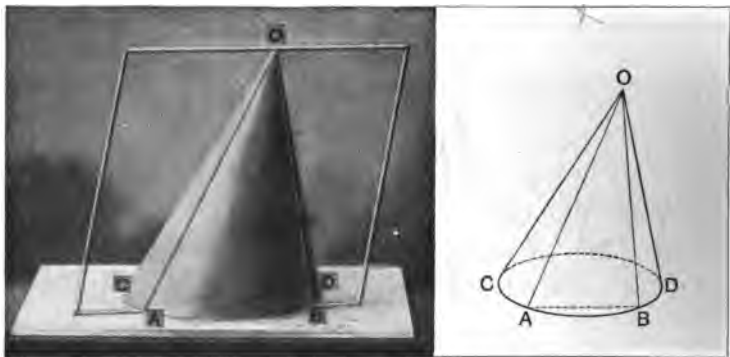
The vertex of the conical surface is called the **vertex** of the cone, and the elements of the conical surface are also called **elements of the cone**.



CONES

## PROPOSITION V. THEOREM

**789.** *Every section of a cone made by a plane passing through its vertex and cutting its base is a triangle.*



**GIVEN**—the cone  $O-CABD$ , whose base is cut in the line  $AB$  by a plane passed through  $O$ .

**TO PROVE**—the section made by this plane is a triangle.

The intersection  $AB$  is a straight line. § 528

We therefore need only to prove that the intersections  $OA$  and  $OB$  are straight.

Draw straight lines from  $O$  to  $A$  and  $B$ .

These straight lines lie in both the cutting plane and the conical surface. §§ 524, 785

Therefore they form the intersections of this plane and the conical surface.

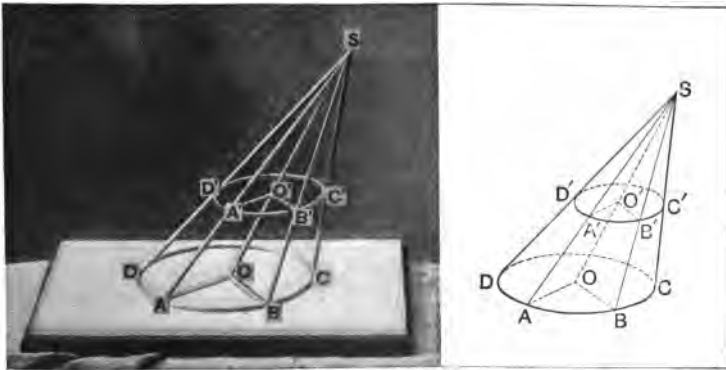
Hence the section made by the plane  $OAB$  is a triangle.

Q. E. D.

**790. Defs.**—A cone whose base is a circle is called a **circular cone**. The straight line joining the vertex of a circular cone to the centre of its base is the **axis** of the cone.

PROPOSITION VI. THEOREM

**791.** *Every section of a circular cone made by a plane parallel to its base is a circle, whose centre is the intersection of the axis with the plane parallel to the base.*



**GIVEN**—the circular cone  $S-ABCD$  of which  $A'B'C'D'$  is a section made by a plane parallel to its base.

Let the axis  $SO$  intersect the plane  $A'B'C'D'$  in  $O'$ .

**TO PROVE**—that  $A'B'C'D'$  is a circle and that  $O'$  is its centre.

Let  $A'$  and  $B'$  be any two points in the perimeter of  $A'B'C'D'$ .

Pass a plane through  $SO$  and  $A'$  and another through  $SO$  and  $B'$ .

Let  $SA$  and  $SB$  be the elements in which these planes intersect the conical surface, and  $AO$ ,  $BO$  and  $A'O'$ ,  $B'O'$  the straight lines in which they cut the parallel planes.

Then  $A'O'$  is parallel to  $AO$ , and  $B'O'$  is parallel to  $BO$ .

If this plane should meet the surface of the cone in any point  $X$ , not in  $AB$ , draw the element  $BY$  passing through  $X$ .

Then  $BY$  would lie in the plane  $BM$ . § 524

Therefore  $AD$  would meet the curve  $AT$  in two points,  $A$  and  $Y$ .

This is contrary to the hypothesis that  $AD$  is tangent to the base.

Hence the plane  $BM$  does not meet the surface of the cone except in  $AB$ .

It is therefore tangent to the cone. § 796

Q. E. D.

**798.** COR. I. *Through a given element, one and only one plane tangent to the cone can be drawn.*

**799.** COR. II. *If a plane is tangent to a cone, its intersection with the plane of the base is tangent to the base.*

**800.** *Exercise.*—Show how to draw a plane through a given point tangent to a cone.

#### THE SPHERE

**801.** *Defs.*—A **spherical surface** is a closed surface all points of which are equidistant from a point called the **centre**.

**802.** *Defs.*—A **sphere** is a solid bounded by a spherical surface.

A **radius** of the sphere is a straight line joining the centre to a point of the surface.

A **diameter** of the sphere is a straight line drawn through the centre and terminated at both ends by the surface.

PROPOSITION VII. THEOREM

**793.** *A right circular cone may be generated by the revolution of a right triangle about one of its sides as an axis.*

The proof is left to the student.

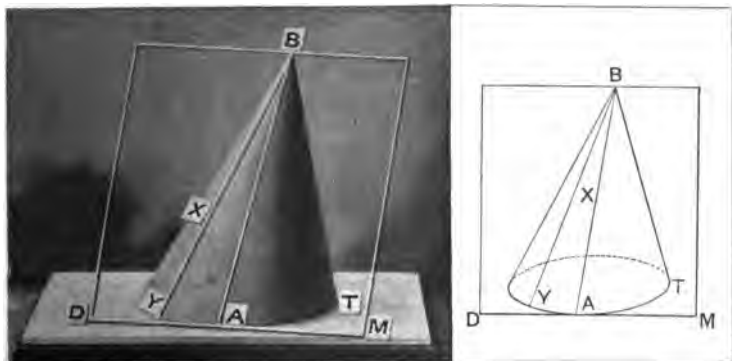
**794.** *Def.*—From its mode of generation a right circular cone is also called a **cone of revolution**.

**795.** *COR.* *The elements of a cone of revolution are all equal.*

**796.** *Def.*—A plane is **tangent** to a cone when it passes through an element and meets its surface in no other point.

PROPOSITION VIII. THEOREM

**797.** *A plane passing through a tangent to the base of a cone and the element drawn to the point of contact is tangent to the cone.*

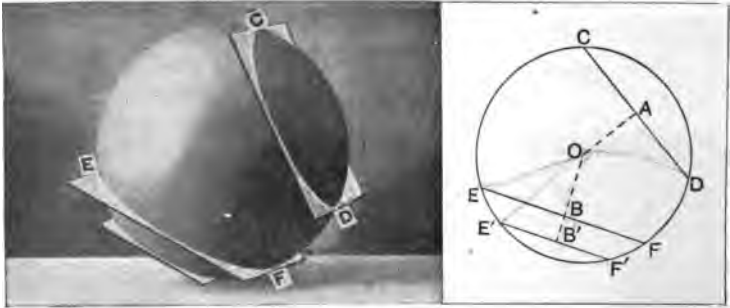


**GIVEN**—the cone  $ABT$ , the tangent  $AD$  to its base, and the element  $AB$  drawn through the point of contact.

**TO PROVE**—that the plane  $BM$ , passing through  $AD$  and  $AB$ , is tangent to the cone.



**805. COR. II.** *Circles of the sphere equidistant from its centre are equal; and conversely.*



*Hint.*—This is proved by dropping perpendiculars  $OA$  and  $OB$  from the centre of the sphere on the planes of the two circles.

We then pass a plane through  $OA$  and  $OB$  intersecting the sphere in the circle  $CDFE$  and the two circles in question in the diameters  $CD$  and  $E'F'$ .

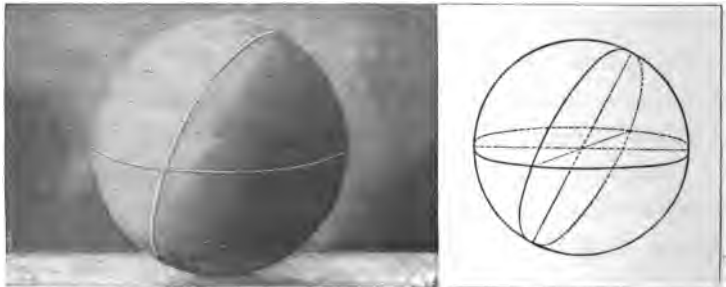
The proof then consists in applying §§ 170, 158.

**806. COR. III.** *The more distant a circle of the sphere is from its centre, the smaller is the circle; and conversely.*

**807. Def.**—A circle whose plane passes through the centre of the sphere is called a **great circle**.

**808. Def.**—A circle whose plane does not pass through the centre of the sphere is called a **small circle**.

**809. COR. IV.** *All great circles are equal.*



**810.** COR. V. *Any two great circles bisect each other.*

*Hint.*—Since they have the same centre, their intersection is a diameter of each circle.

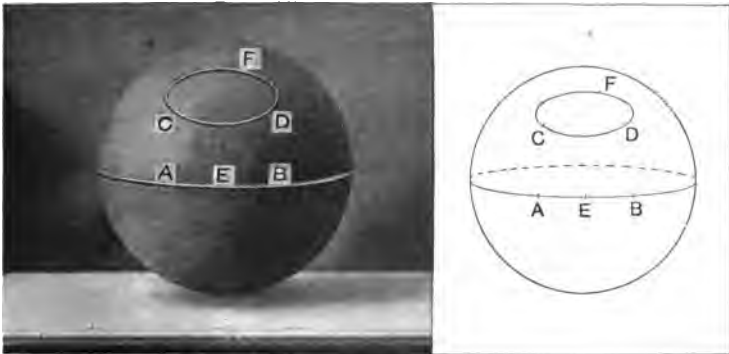
**811.** COR. VI. *Every great circle divides the sphere and its surface into two equal parts.*

*Hint.*—Prove by superposition.

**812.** COR. VII. *Through any three points on the surface of a sphere one and only one circle can be drawn.*

**813.** COR. VIII. *Through any two points on the surface of a sphere, not at the extremities of a diameter, one and only one great circle can be drawn.*

*Hint.*—The two points together with the centre of the sphere determine the plane of a great circle.



*Question.*—If the two points are at the extremities of a diameter, how is Corollary VIII. modified?

**814.** *Def.*—By the distance between two points on the surface of a sphere is usually meant the arc of a great circle, less than a semi-circumference, joining them.

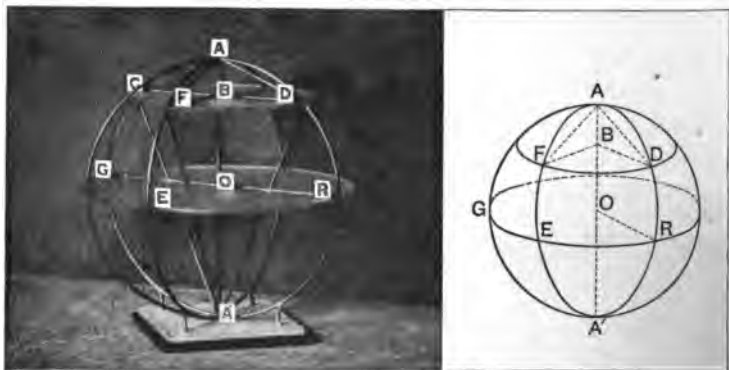
Thus the distance between the points *A* and *B* is the arc *AEB*.

**815.** *Def.*—The diameter of a sphere which is perpendicular to the plane of a circle of the sphere is called the axis of that circle.

**816. Def.**—The **poles** of a circle are the extremities of its axis.

PROPOSITION X. THEOREM

**817.** *All points in the circumference of a circle of the sphere are equally distant from each of its poles.*



**GIVEN**—any two points  $F$  and  $D$  in the circumference of a circle  $CFD$  and  $A$  and  $A'$ , the poles of  $CFD$ .

Draw the great-circle arcs  $AF$ ,  $AD$ ,  $A'F$ ,  $A'D$ .

**TO PROVE** arc  $AF = \text{arc } AD$ , and arc  $A'F = \text{arc } A'D$ .

Let  $B$  be the intersection of the axis  $AA'$  with the plane of  $CFD$ . Draw the straight lines  $AF$  and  $AD$ .

Now  $BF = BD$ . § 803

Hence chord  $AF = \text{chord } AD$ . § 539 I

Therefore arc  $AF = \text{arc } AD$ . § 164

Similarly we may prove

arc  $A'F = \text{arc } A'D$ . Q. E. D.

**818. Def.**—The **polar distance** of a circle of a sphere is the arc of a great circle drawn from its nearer pole to any point of its circumference.

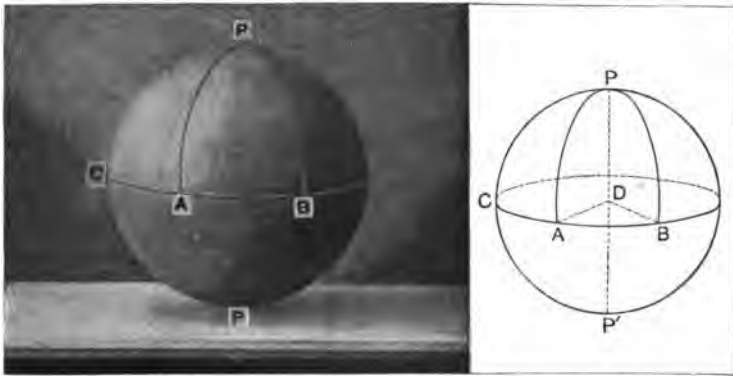
**819. COR.** *The polar distance of a great circle is a quadrant of a great circle.*

*Hint.*—Let  $GER$  be a great circle. Then its centre  $O$  is also the centre of the great circle  $ARA'$ . Hence the arc  $AR$  measures the right angle  $AOR$ .

**820. Def.**—The term **quadrant** in connection with a sphere is used to signify a quadrant of a great circle.

PROPOSITION XI. THEOREM

**821.** *If a point on the surface of a sphere is at a quadrant's distance from two points on that surface, it is the pole of the great circle passed through those points.*



**GIVEN**—a point  $P$  on the surface of a sphere at a quadrant's distance from each of the points  $A$  and  $B$  on that surface.

**TO PROVE** that  $P$  is the pole of the great circle  $AB$ .

Draw the radii  $DP$ ,  $DA$ , and  $DB$ .

Since  $PA$  and  $PB$  are quadrants,  $PDA$  and  $PDB$  are right angles. §§ 804, 194

Therefore  $PD$  is perpendicular to the plane  $DAB$ . § 531

That is,  $P$  is the pole of the great circle  $AB$ . § 816

Q. E. D.

**822. Remark.**—The preceding theorems enable us to draw circumferences upon the surface of a sphere as easily as upon a plane. A pair of compasses with curved branches is employed. The opening of the compasses (distance between their points) is made equal to the chord of the polar distance of the required circle. Then, one point of the compasses being placed at the pole, the other describes the circumference.

If we wish to draw an arc of a *great* circle, the opening of the compasses must be equal to the chord of a quadrant. This can be found when the diameter is known. A method for finding the diameter will be given in the next proposition.

If it is desired to draw an arc of a *great* circle *through two points* on the surface, it is necessary first to find the pole of this circle. For this purpose draw circumferences of great circles with the two points as poles. These two circumferences will intersect in two points, either of which is the required pole. Then the circumference can be drawn as described above.

#### PROPOSITION XII. PROBLEM

**823.** *To find the diameter of a given sphere.*

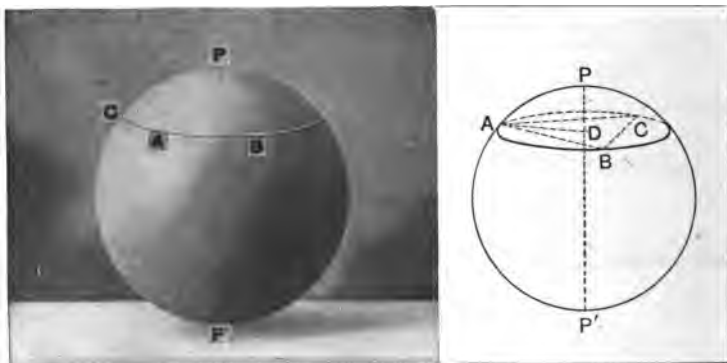


FIG. 1

We suppose the given sphere a material one, and that only measurements on its surface are possible.

*First*, with any point  $P$  on the surface as a pole, and with any opening of the compasses  $AP$ , draw a circumference  $ABC$  on the surface (Fig. 1). Then the straight line  $AP$  is known.

Take any three points  $A, B, C$  in this circumference. Measure with the compasses the straight lines  $AB, BC, CA$ .

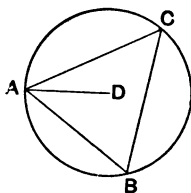


FIG. 2

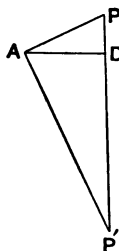


FIG. 3

Secondly, on a plane construct a triangle having  $AB, BC, CA$  as sides (Fig. 2). § 90

Find the centre  $D$  of the circle circumscribing  $ABC$ . § 219

Then the straight line  $AD$  is known.

Thirdly, with  $AD$  as a side and  $AP$  as the hypotenuse, construct the right triangle  $ADP$  (Fig. 3).

Draw  $AP'$  perpendicular to  $AP$ , meeting  $PD$  produced in  $P'$ .

Then  $PP'$  is equal to the diameter of the given sphere.

*Proof.*—In Fig. 1 draw  $PP'$  the axis of the circle  $ABC$  meeting the plane of  $ABC$  in  $D$ . Then  $D$  is the centre of the circle  $ABC$ . § 803

Draw  $DA$  and  $P'A$ .

The triangle  $ABC$  (Fig. 1) equals the triangle  $ABC$  (Fig. 2). § 89

Hence  $AD$  is the same in Figs. 1, 2, and 3. § 158

Now in Fig. 1 the angle  $PDA$  is right. § 530

And  $AP$  is the same in Figs. 1 and 3. Cons.

Hence the right triangles  $ADP$  are equal in Figs. 1 and 3. § 101

Again in Fig. 1 the angle  $PAP'$  is right. § 202

Hence the right triangles  $PAP'$  are equal in Figs. 1 and 3. § 86

Therefore  $PP'$  in Fig. 3 is equal to the diameter of the given sphere.

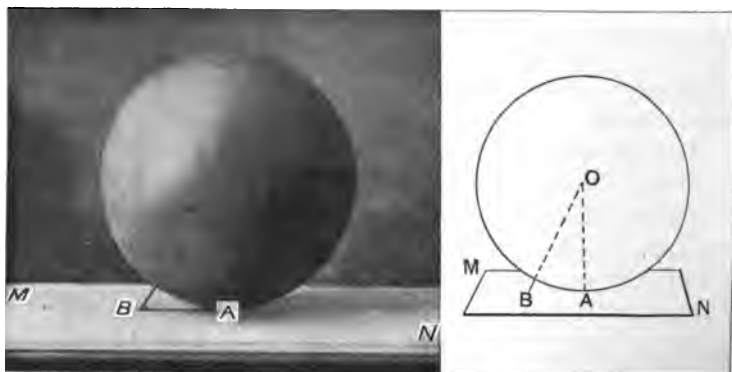
Q. E. F.

**824. Defs.**—A plane is **tangent** to a sphere when it has one, and only one, point in common with the surface of the sphere. This point is called the **point of tangency**.

In the same case the sphere is said to be tangent to the plane.

## PROPOSITION XIII. THEOREM

**825.** *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere; conversely, a plane tangent to a sphere is perpendicular to the radius drawn to the point of tangency.*



**GIVEN**—the plane  $MN$  perpendicular to the radius  $OA$  of the sphere whose centre is  $O$  at its extremity  $A$ .

**TO PROVE** that  $MN$  is tangent to the sphere.

Let  $B$  be *any* point in  $MN$  other than  $A$ . Join  $OB$ .

Then  $OB > OA$ . § 536

Hence  $B$  is outside the sphere. § 801

That is,  $MN$  has only one point  $A$  in common with the surface of the sphere.

Therefore  $MN$  is tangent to the sphere. § 824

**Q. E. D.**

**CONVERSELY :**

**GIVEN**—the plane  $MN$  tangent to the sphere whose centre is  $O$ .

Draw the radius  $OA$  to the point of tangency.

**TO PROVE** that  $MN$  is perpendicular to  $OA$ .

Let  $B$  be any point in  $MN$  other than  $A$ . Join  $OB$ .

Since  $MN$  is tangent to the sphere at  $A$ ,  $B$  lies outside of the sphere. § 824

Hence  $OB > OA$ .

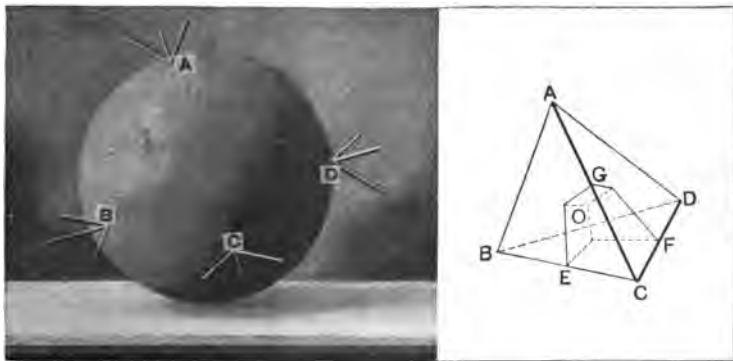
That is,  $OA$  is the shortest line from  $O$  to  $MN$ .

Therefore  $MN$  is perpendicular to  $OA$ . § 536  
Q. E. D.

**826. Exercise.**—Prove that three planes perpendicular respectively to the three edges of a triedral angle meet in a point.

PROPOSITION XIV. THEOREM

**827.** *A spherical surface can be passed through any four points, not in the same plane, and but one.*

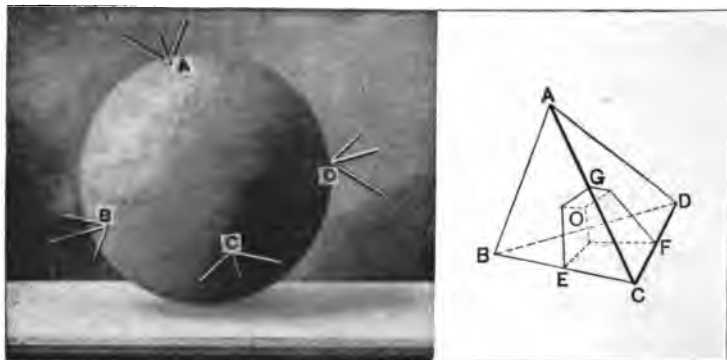


**GIVEN** the four points  $A, B, C, D$ , not in the same plane.

**TO PROVE**—that one, and only one, spherical surface can be passed through these points.

Form a tetrahedron having these points as vertices.





Draw three planes  $EO$ ,  $FO$ , and  $GO$  perpendicular respectively to the edges  $BC$ ,  $CD$ , and  $CA$  at their middle points.

The plane  $EO$  is the locus of points equidistant from  $B$  and  $C$ ; the plane  $FO$  is the locus of points equidistant from  $C$  and  $D$ ; and the plane  $GO$  is the locus of points equidistant from  $C$  and  $A$ .

§ 611

Hence the intersection  $O$  of these three planes is equidistant from  $A$ ,  $B$ ,  $C$ , and  $D$ , and is the only point equidistant from those points.

§ 102

Therefore the spherical surface described with  $O$  as a centre, and the line  $OA$  as a radius, will pass through the four points, and will be the only spherical surface that can be passed through the four points.

Q. E. D.

**828.** COR. I. *The six planes perpendicular to the six edges of a tetrahedron at their middle points meet in a point.*

**829.** COR. II. *The four straight lines perpendicular to the faces of a tetrahedron at the centres of their circumscribing circles meet in a point.*

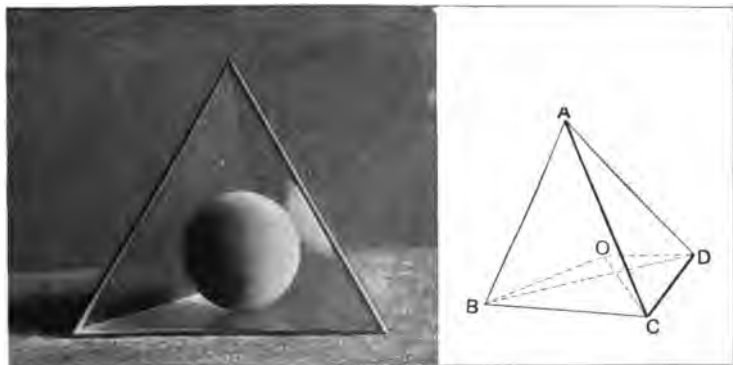
**830.** *Exercise.*—Prove that the three planes bisecting

the dihedral angles at the base of a tetraedron meet in a point.

**831.** *Def.*—A sphere is **inscribed** in a polyedron when its centre is within the polyedron and its surface is tangent to all the faces of the polyedron.

PROPOSITION XV. THEOREM

**832.** *A sphere can be inscribed in any tetraedron, and but one.*



**GIVEN** the tetraedron  $ABCD$ .

**TO PROVE**—that one, and only one, sphere can be inscribed in it.

Bisect the dihedral angles  $BC$ ,  $CD$ , and  $DB$  by the planes  $BOC$ ,  $COD$ , and  $DOB$ .

The plane  $BOC$  is the locus of points equidistant from the faces  $BCD$  and  $BAC$ ; the plane  $COD$  is the locus of points equidistant from the faces  $BCD$  and  $CAD$ ; and the plane  $DOB$  is the locus of points equidistant from the faces  $BCD$  and  $DAB$ .

Hence the intersection  $O$  of these three planes is equidistant from the four faces of the tetraedron, and is the only point equidistant from the faces.

Therefore the sphere described with  $O$  as a centre, and the perpendicular distance from  $O$  upon the face  $BCD$  as a radius, will be tangent to all the faces of the tetraedron and hence will be inscribed in the tetraedron. § 825

And it will be the only sphere that can be inscribed in the tetraedron. Q. E. D.

**833. COR.** *The six planes bisecting the six dihedral angles of a tetraedron meet in a point.*

#### SPHERICAL ANGLES

**834. Def.**—The angle of two curves meeting in a common point is the angle formed by the two tangents to the curves at that point.

**835. Def.**—A spherical angle is the angle between two intersecting arcs of great circles on the surface of a sphere.

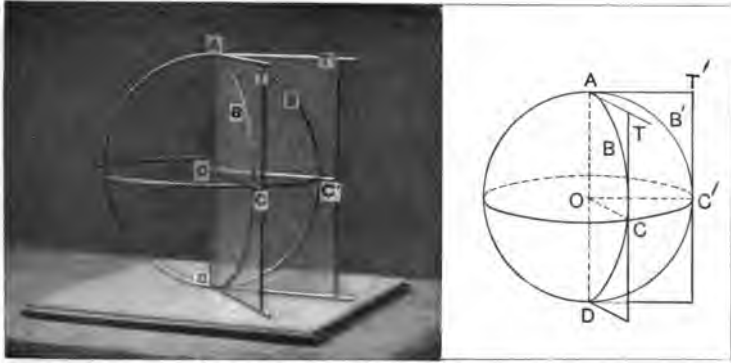
#### PROPOSITION XVI. THEOREM

**836.** *The angle of two arcs of great circles on a spherical surface is*

- I. *Equal to the plane angle of the dihedral angle formed by their planes.*
- II. *Measured by the arc of a great circle described with its vertex as a pole and included between its sides, produced if necessary.*

GIVEN— $AB$  and  $AB'$ , two arcs of great circles whose planes form a dihedral angle having the diameter  $AD$  for edge:

With  $A$  as a pole describe a great circle cutting  $AB$  and  $AB'$ , produced, if necessary, in  $C$  and  $C'$ .



I. TO PROVE—the angle  $BAB'$  is equal to the plane angle of the diedral angle  $BADB'$ .

Draw  $AT$  and  $AT'$  tangent to the arcs  $AB$  and  $AB'$  respectively.

Then by definition the angles  $BAB'$  and  $TAT'$  are identical. § 834

But  $AT$  and  $AT'$  are perpendicular to  $OA$ . § 173

Hence  $TAT'$ , or  $BAB'$ , is the plane angle of the diedral angle  $BADB'$ . § 567

Q. E. D.

II. TO PROVE—that the angle  $BAB'$  is measured by the arc  $CC'$ .

Join the centre of the sphere,  $O$ , to  $C$  and  $C'$ .

Then, since  $A$  is the pole of  $CC'$ , the plane  $COC'$  is perpendicular to  $AO$ . § 816

Hence  $COC'$  is the plane angle of the diedral angle  $BADB'$ . § 530

Therefore the angle  $BAB'$  is equal to the angle  $COC'$ .

But  $COC'$  is measured by the arc  $CC'$ . § 191

Therefore  $BAB'$  is measured by the arc  $CC'$ . Q. E. D.

**837. COR. I.** Any great-circle-arc  $AC$ , drawn through the pole of a given great circle  $CC'$  is perpendicular to the circumference  $CC'$ .

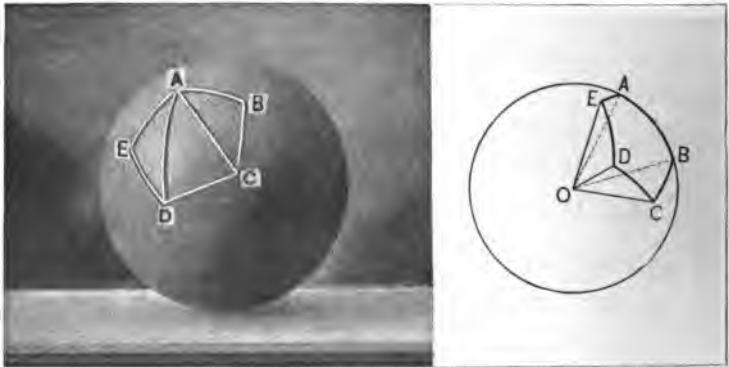
*Hint.*—First prove the plane  $AOC$  perpendicular to the plane  $COC'$ .  
Then the plane angle of the dihedral angle  $OC$  is a right angle.

**838. COR. II.** Conversely, any great-circle-arc perpendicular to a given arc must pass through the pole of the given arc.

*Hint.*—Apply § 576.

### SPHERICAL POLYGONS

**839. Defs.**—A **spherical polygon** is a portion of a spherical surface bounded by three or more arcs of great circles; as  $ABCDE$ .



The bounding arcs are called the **sides** of the spherical polygon; their intersections, the **vertices**, and the angles formed by the sides at the vertices, the **angles** of the spherical polygon.

**840. Def.**—A **diagonal** of a spherical polygon is an arc of a great circle joining any two vertices not consecutive.

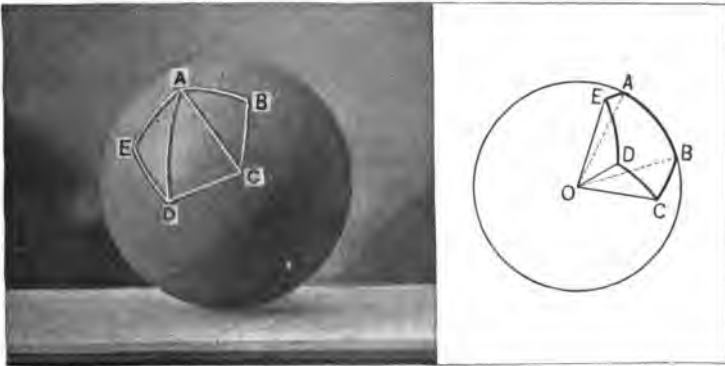
**841. Remark.**—The sides of a spherical polygon are usually measured in degrees.

**842. Def.**—The polyedral angle, whose vertex is at the centre of the sphere, formed by the planes of the sides of a spherical polygon, is said to **correspond** to the spherical polygon.

Thus the polyedral angle  $O - ABCDE$  corresponds to the spherical polygon  $ABCDE$ .

PROPOSITION XVII. THEOREM

**843.** *The sides of a spherical polygon measure the corresponding face angles of the corresponding polyedral angle ; and its angles are equal to the plane angles of the corresponding dihedral angles.*



*Hint.*—This proposition is an immediate consequence of §§ 191, 836 I.

**844. Remark.**—Since each face angle of a polyedral angle is assumed to be less than two right angles, each side of a spherical polygon will be assumed to be less than a semi-circumference.

**845. Def.**—The **parts** of a spherical polygon are its sides and angles.

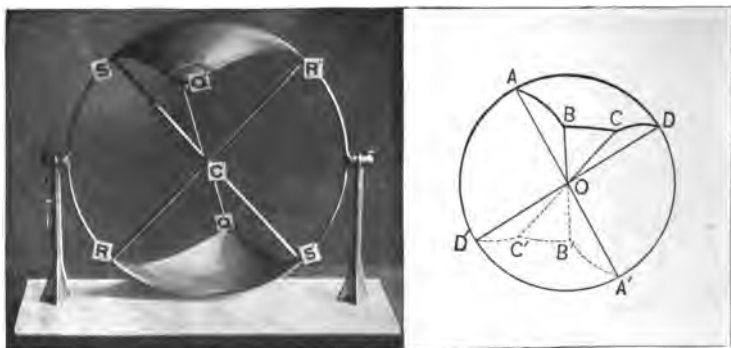
**846. Remark.**—By means of the relations between the parts of a spherical polygon and the parts of its corresponding polyedral angle we can, from any property of polyedral angles, deduce an analogous property of spherical polygons.

Reciprocally, from any property of spherical polygons, we can infer an analogous property of polyedral angles.

**847. Defs.**—A **spherical triangle** is a spherical polygon of three sides. It is called **isosceles**, **equilateral**, or **right-angled** in the same cases in which a plane triangle would be so named.

#### SYMMETRICAL SPHERICAL TRIANGLES AND POLYGONS

**848. Def.**—Two spherical polygons are **vertical** when their vertices are situated by pairs at opposite ends of the same diameter.



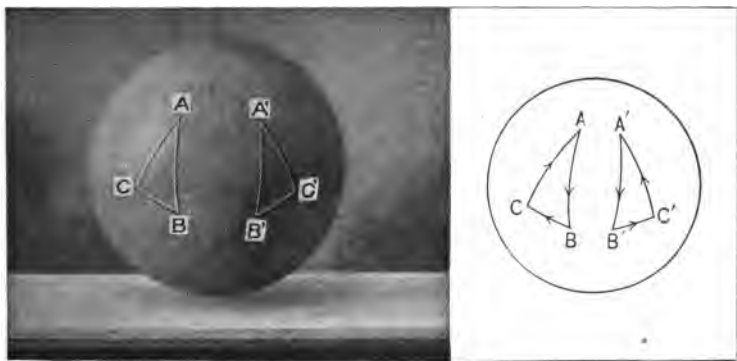
Thus, to determine the spherical polygon vertical to  $ABCD$  we draw the diameters  $AOA'$ ,  $BOB'$ ,  $COC'$ ,  $DOD'$ . Then  $A'B'C'D'$  is vertical to  $ABCD$ .

**849. THEOREM.** *Two spherical polygons are vertical, if their corresponding polyedral angles are vertical, and conversely.*

*Hint.*—This follows immediately from the preceding definition and § 599.

**850. Def.**—Two spherical polygons are **symmetrical** when they have the same number of parts equal each to each and arranged in opposite order.

Thus, in the triangles  $ABC$  and  $A'B'C'$ , if  $A=A'$ ,  $B=B'$ ,  $C=C'$ ,  $AB=A'B'$ ,  $BC=B'C'$ ,  $CA=C'A'$ , and the order of arrangement of the parts is opposite in the two figures, the triangles are symmetrical.



The meaning of the words “arranged in opposite order” will be made clearer by the following explanation :

In the figure above the direction of motion in going from  $A$  to  $B$  to  $C$  to  $A$  is the direction of rotation of the hands of a clock ; the direction of motion in going from  $A'$  to  $B'$  to  $C'$  to  $A'$  is opposite to the direction of rotation of the hands of a clock ; supposing that in each case we look at the surface of the sphere from the outside. If we look at the surface from the inside, the directions will be reversed.

**851. THEOREM.** *Two spherical polygons are symmetrical, if their corresponding polyedral angles are symmetrical, and conversely.*

This follows immediately from the preceding definition and §§ 600, 843.



## PROPOSITION XVIII. THEOREM

**852.** *Two vertical spherical polygons are symmetrical.*

*Proof.*—The corresponding polyedral angles at the centre are vertical. § 849

They are therefore symmetrical. § 601

Hence the spherical polygons are symmetrical. § 851

Q. E. D.

## PROPOSITION XIX. THEOREM

**853.** *Of two symmetrical spherical polygons either is equal to the vertical of the other.*

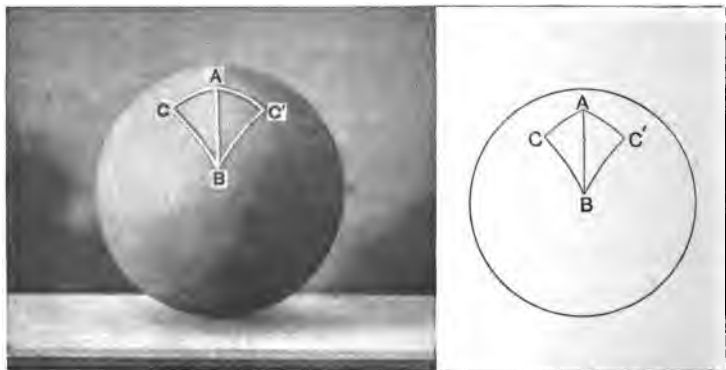
*Proof.*—The corresponding polyedral angles at the centre are symmetrical. § 851

Hence either may be made to coincide with the vertical of the other. § 602

When this is done, the two spherical polygons will be vertically opposite. § 849

Q. E. D.

**854.** *Remark.*—In general two symmetrical spherical polygons cannot be made to coincide, and hence are not equal.



Thus, if two symmetrical spherical triangles  $ABC$  and  $A'B'C'$  are not isosceles, the only side of  $A'B'C'$  with which  $AB$  can be made to coincide is  $A'B'$ . If we place  $A$  upon  $A'$  and  $B$  upon  $B'$ ,  $C$  and  $C'$  will fall on opposite sides of  $AB$ . If we place  $A$  upon  $B'$  and  $B$  upon  $A'$ ,  $C$  and  $C'$  will fall on the same side of  $AB$ , but will not coincide. But if the triangles are *isosceles*, they can be made to coincide, as the following proposition will show.

PROPOSITION XX. THEOREM

**855.** *Two symmetrical isosceles spherical triangles are equal.*



*Hint.*—Show that the corresponding triedral angles have two face angles and the included dihedral angle respectively equal, and similarly arranged ( $A$  corresponding to  $A'$ , but  $B$  to  $C'$  and  $C$  to  $B'$ ). Then superpose these triedral angles (§ 595).

**856.** COR. I. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*

*Hint.*—In superposing the symmetrical isosceles triangles in the above figure, the angle  $B'$  is made to coincide with  $C$ . But we know that  $B' = B$ .

**857.** COR. II. *If a spherical triangle is equilateral, it is also equiangular.*

**858.** COR. III. *If two face angles of a triedral angle are equal, the opposite dihedral angles are equal.*

**859.** COR. IV. *If the three face angles of a triedral angle are equal, its three dihedral angles are equal.*

## PROPOSITION XXI. THEOREM

**860.** *If two angles of a spherical triangle are equal, the opposite sides are equal.*

*Hint.*—Form the symmetrical triangle. Show that the corresponding triedral angles have a face angle and the adjacent diedral angles respectively equal, and similarly arranged. Then superpose these triedral angles (§ 596).

**861.** COR. I. *If a spherical triangle is equiangular, it is also equilateral.*

**862.** COR. II. *If two diedral angles of a triedral angle are equal, the opposite face angles are equal.*

**863.** COR. III. *If the three diedral angles of a triedral angle are equal, the three face angles are equal.*

## PROPOSITION XXII. THEOREM

**864.** *Any side of a spherical triangle is less than the sum of the two others.*

*Hint.*—Form the corresponding triedral angle.  
Then apply §§ 843, 593.

**865.** COR. I. *Any side of a spherical polygon is less than the sum of all the others.*

*Hint.*—Divide the polygon into triangles by diagonals from any vertex.

**866.** COR. II. *Any face angle of a polyedral angle is less than the sum of all the others.*

**867.** *Def.*—A spherical polygon is **convex** when its corresponding polyedral angle is convex.

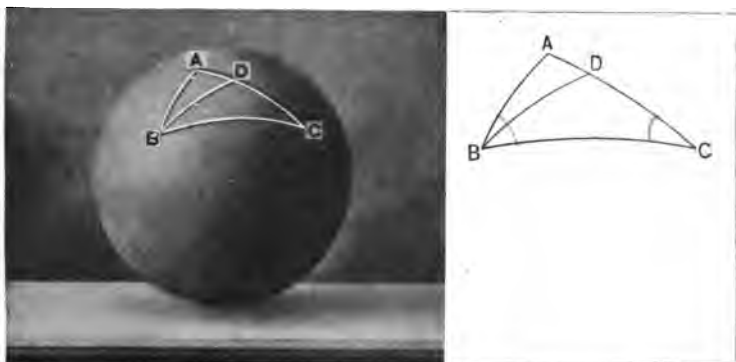
## PROPOSITION XXIII. THEOREM

**868.** *The sum of the sides of a convex spherical polygon is less than the circumference of a great circle.*

*Hint.*—Form the corresponding polyedral angle.  
Then apply §§ 843, 594.

PROPOSITION XXIV. THEOREM

**869.** *If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.*



**GIVEN** the spherical triangle  $ABC$  in which angle  $ABC > ACB$ .

**TO PROVE** side  $AC > AB$ .

Draw  $BD$  making angle  $DBC = DCB$ .

Then  $DC = DB$ . § 860

Adding  $AD$  to each of these equals we have

$$AC = AD + DB.$$

But  $AD + DB > AB$ . § 864

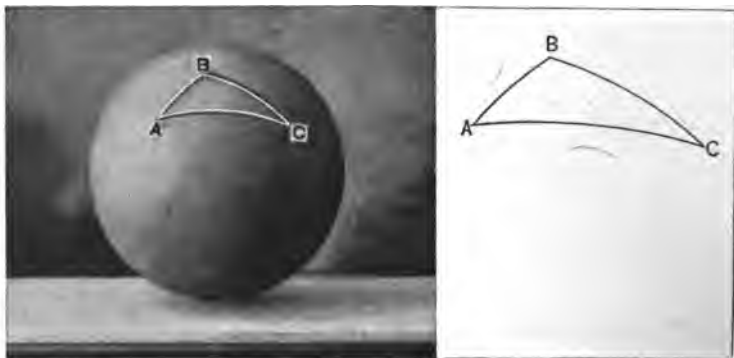
Therefore  $AC > AB$ . Q. E. D.

**870. COR.** *If two dihedral angles of a trihedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater dihedral angle.*

## PROPOSITION XXV. THEOREM

**871.** *If two sides of a spherical triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.*

[Converse of Proposition XXIV.]



**GIVEN** the spherical triangle  $ABC$  in which side  $AC > AB$ .

**TO PROVE** angle  $ABC > ACB$ .

If  $ABC$  were equal to  $ACB$ , then  $AC$  would equal  $AB$ .

§ 860

If  $ABC$  were less than  $ACB$ , then  $AC$  would be less than  $AB$ .

§ 869

Both these conclusions are contrary to the hypothesis.

Therefore  $ABC > ACB$ .

Q. E. D.

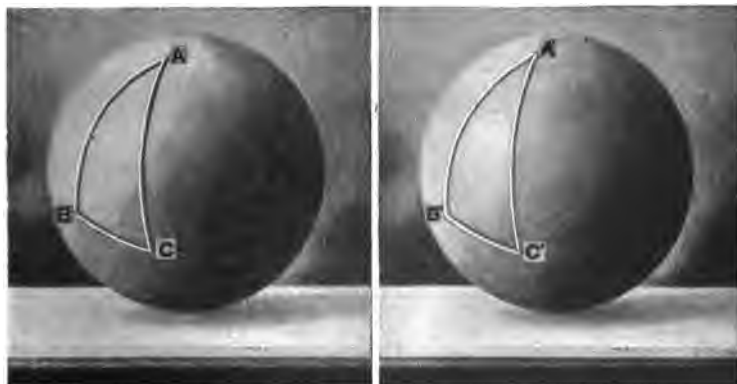
**872. COR.** *If two face angles of a trihedral angle are unequal, the opposite dihedral angles are unequal, and the greater dihedral angle is opposite the greater face angle.*

## PROPOSITION XXVI. THEOREM

**873.** *Two triangles on the same sphere are equal:*

- I. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*
- II. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*
- III. *If the three sides of one are equal respectively to the three sides of the other.*

*Provided in each case that the parts given equal are arranged in the same order in both triangles.*



*Proof.*—In each case the corresponding triedral angles are equal. §§ 595, 596, 597

They can therefore be placed in coincidence.

At the same time the triangles coincide.

Therefore the two given triangles are equal.

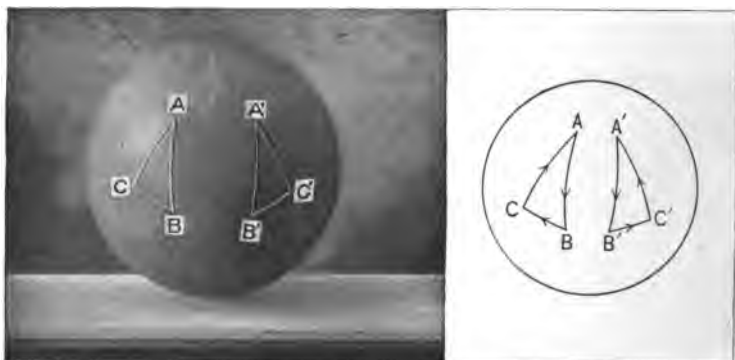
Q. E. D.

## PROPOSITION XXVII. THEOREM

**874.** *Two triangles on the same sphere are symmetrical :*

- I. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*
- II. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*
- III. *If the three sides of one are equal respectively to the three sides of the other.*

*Provided in each case that the parts given equal are arranged in opposite order in the two triangles.*

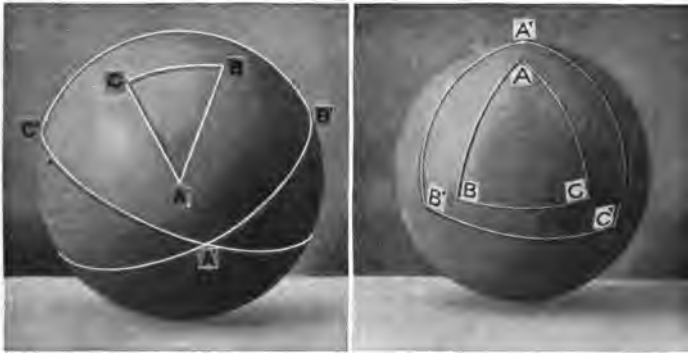


*Proof.*—In each case the corresponding triedral angles at the centre are symmetrical. § 603

Therefore the two given triangles are symmetrical. § 851  
Q. E. D.

## POLAR TRIANGLES

**875.** *Def.*—If, with the vertices of a spherical triangle as poles, arcs of great circles are described, these arcs will divide the spherical surface into eight triangles. One of these is called the **polar triangle** of the given triangle.



The method of selecting the polar triangle from the eight is as follows: Call the given triangle  $ABC$  and the polar triangle  $A'B'C'$ . Then  $A'$  is one of the intersections of the arcs described from  $B$  and  $C$  as poles; that one which is less than a quadrant's distance from  $A$ . In a similar way  $B'$  and  $C'$  are determined.

PROPOSITION XXVIII. THEOREM

**876.** *If one spherical triangle is the polar triangle of another, then, reciprocally, the second spherical triangle is the polar triangle of the first.*

GIVEN that  $A'B'C'$  is the polar triangle of  $ABC$ .

TO PROVE that  $ABC$  is the polar triangle of  $A'B'C'$ .

Since  $B$  is the pole of  $A'C'$ , the distance  $A'B$  is a quadrant; since  $C$  is the pole of  $A'B'$ , the distance  $A'C$  is a quadrant. § 819

Therefore  $A'$  is the pole of  $BC$ . § 821

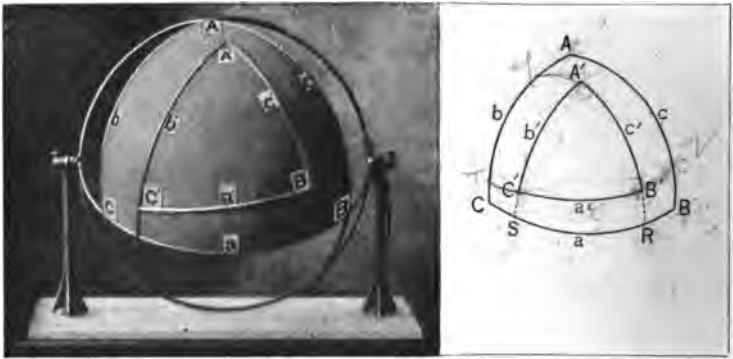
Similarly,  $B'$  is the pole of  $CA$ , and  $C'$  is the pole of  $AB$ .

Since also the distances  $AA'$ ,  $BB'$ , and  $CC'$  are each less than a quadrant,  $ABC$  is the polar triangle of  $A'B'C'$ . § 875  
Q. E. D.



## PROPOSITION XXIX. THEOREM

**877.** *In two polar triangles, each angle of one is measured by the supplement of the side of which its vertex is the pole in the other.*



**GIVEN**—the polar triangles  $ABC$  and  $A'B'C'$ . Let  $A, B, C$ , and  $A', B', C'$  denote their angles, measured in degrees, and  $a, b, c$ , and  $a', b', c'$  the sides respectively opposite these angles, also measured in degrees.

**TO PROVE**— $A' + a = 180^\circ$ ,  $B' + b = 180^\circ$ ,  $C' + c = 180^\circ$ ,  
 $A + a' = 180^\circ$ ,  $B + b' = 180^\circ$ ,  $C + c' = 180^\circ$ .

Produce  $A'B'$  and  $A'C'$  to meet  $BC$  at  $R$  and  $S$ .

Then, since  $B$  is the pole of  $A'S$  and  $C$  the pole of  $A'R$ ,  
 $BS$  and  $CR$  are quadrants. § 819

Therefore  $BS + CR = 180^\circ$ ,  
 or  $BR + RS + RS + SC = 180^\circ$ ,  
 or  $RS + BC = 180^\circ$ .

But  $BC = a$ , and  $RS$  measures the angle  $A'$ . § 836 II

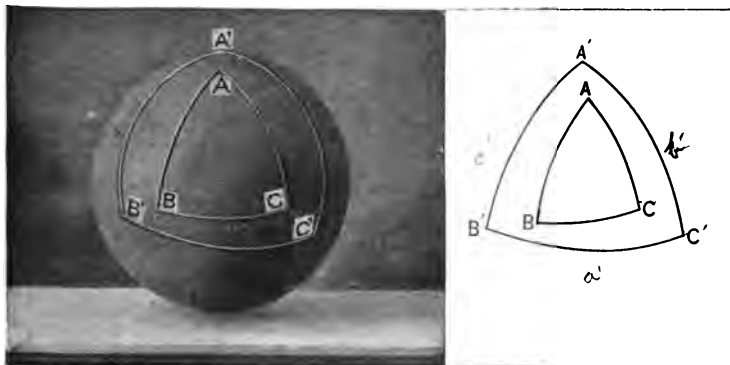
Therefore  $A' + a = 180^\circ$ .

To prove the relation  $A + a' = 180^\circ$  we would produce  $B'C'$  to meet  $AB$  and  $AC$ .

In a similar manner the remaining relations are proved. Q. E. D.

PROPOSITION XXX. THEOREM

**878.** *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



GIVEN the spherical triangle  $ABC$ .

Denote its angles by  $A, B, C$ , and the sides opposite in the polar triangle by  $a', b', c'$ .

TO PROVE  $A + B + C > 180^\circ$  and  $< 540^\circ$ .

We have

$$\begin{aligned} A &= 180^\circ - a' \\ B &= 180^\circ - b' \\ C &= 180^\circ - c'. \end{aligned}$$

§ 877

Adding these equations we get

$$A + B + C = 540^\circ - (a' + b' + c').$$

Hence  $A + B + C < 540^\circ$ .

Q. E. D.

Also, since  $a' + b' + c' < 360^\circ$ ,

§ 868

13°  $A + B + C > 180^\circ$ .

Q. E. D.

**879.** COR. I. *A spherical triangle may have two, or even three, right angles; also two, or even three, obtuse angles.*

**880.** Defs.—A spherical triangle having two right angles is called a **bi-rectangular triangle**.

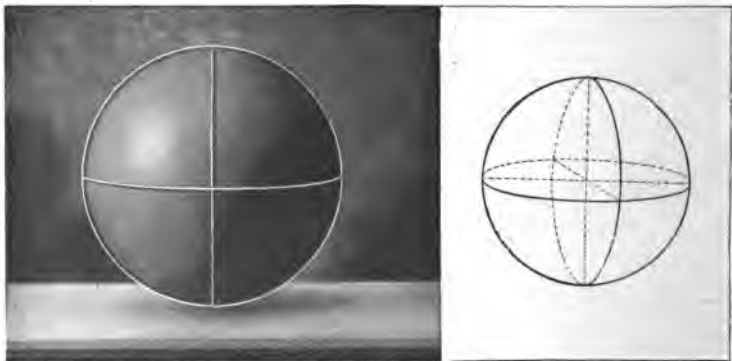
A spherical triangle having three right angles is called a **tri-rectangular triangle**.

**881.** COR. II. *In a bi-rectangular triangle the sides opposite the right angles are quadrants.*



*Hint.*—Apply §§ 838, 819.

**882.** COR. III. *Three planes passed through the centre of a sphere, each perpendicular to the other two, divide the surface of the sphere into eight equal tri-rectangular triangles.*

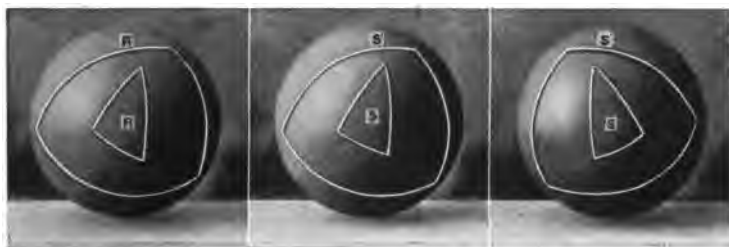


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PROPOSITION XXXI. THEOREM

**883.** *If two triangles on the same sphere are mutually equiangular :*

- I. They are equal, when the equal angles are arranged in the same order in both triangles.*
- II. They are symmetrical, when the equal angles are arranged in opposite order in the two triangles.*



GIVEN—two mutually equiangular spherical triangles  $R$  and  $S$ .

TO PROVE—that  $R$  and  $S$  are either equal or symmetrical.

Let  $R'$  and  $S'$  be the polar triangles of  $R$  and  $S$  respectively.

Then, since  $R$  and  $S$  are mutually equiangular, we can show by means of the relations proved in Proposition XXIX. that  $R'$  and  $S'$  are mutually equilateral.

Hence  $R'$  and  $S'$  are either equal or symmetrical.

§§ 873 III, 874 III

They are therefore mutually equiangular. § 850

Hence we can show that  $R$  and  $S$ , the polar triangles of  $R'$  and  $S'$ , are mutually equilateral.

Therefore  $R$  and  $S$  are equal or symmetrical, according to the arrangement of their homologous parts.

§§ 873 III, 874 III

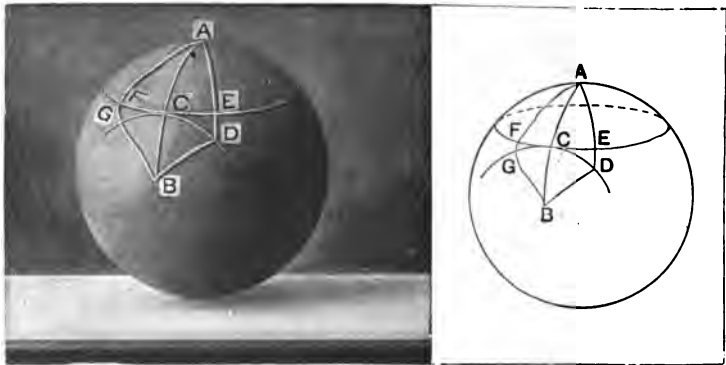
Q. E. D.

**884.** COR. *If two triedral angles have their dihedral angles equal each to each :*

- I. *They are equal, when the equal dihedral angles are arranged in the same order in both triedral angles.*
- II. *They are symmetrical, when the equal dihedral angles are arranged in opposite order in the two triedral angles.*

PROPOSITION XXXII. THEOREM

**885.** *The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle, not greater than a semi-circumference, joining those points.*



GIVEN—an arc of a great circle  $AB$ , not greater than a semi-circumference, joining the points  $A$  and  $B$  on a spherical surface.

TO PROVE—that  $AB$  is the shortest line that can be drawn on the surface between  $A$  and  $B$ .

CASE I. *When  $AB$  is less than a semi-circumference.*

Let  $C$  be any point of  $AB$ .

With  $A$  and  $B$  as poles describe circumferences whose polar distances are  $AC$  and  $BC$ .

These circumferences have only the point  $C$  in common.

For, let  $D$  be any other point of the circumference whose pole is  $B$ .

Draw the great-circle-arcs  $AD$  and  $BD$  and let  $AD$  meet the circumference whose pole is  $A$  in  $E$ .

Then  $AD + BD > AC + BC$ . § 864

But  $BD = BC$  and  $AC = AE$ . § 817

Hence  $AD > AE$ .

Therefore  $D$  lies outside the small circle whose pole is  $A$ , and the two small circles have only the point  $C$  in common.

Now we will prove that the shortest line on the surface between  $A$  and  $B$  must pass through  $C$ .

Let  $AFGB$  be any line on the surface between  $A$  and  $B$  that does not pass through  $C$ .

It must cut the small circles in separate points  $F$  and  $G$ .

Now, whatever may be the nature of the line  $AF$ , an equal line can be drawn on the surface between  $A$  and  $C$ .

[This can be shown by supposing the spherical surface to revolve on the axis of the small circle  $FCE$ , so that  $F$  will move along the small circle to  $C$ , while  $A$  remains fixed.]

Similarly a line equal to  $BG$  can be drawn from  $B$  to  $C$ .

There will then lie between  $A$  and  $B$  and passing through  $C$  a line less than  $AFGB$  by the portion  $FG$ .

We have now proved that through  $C$  can be drawn a line joining  $A$  and  $B$  less than any line joining  $A$  and  $B$  that does not pass through  $C$ .

Hence the shortest line must pass through  $C$ .

But  $C$  is any point in the arc  $AB$ .

Therefore the shortest line between  $A$  and  $B$  must pass through every point of the arc  $AB$  and hence must coincide with that arc.

Q. E. D.

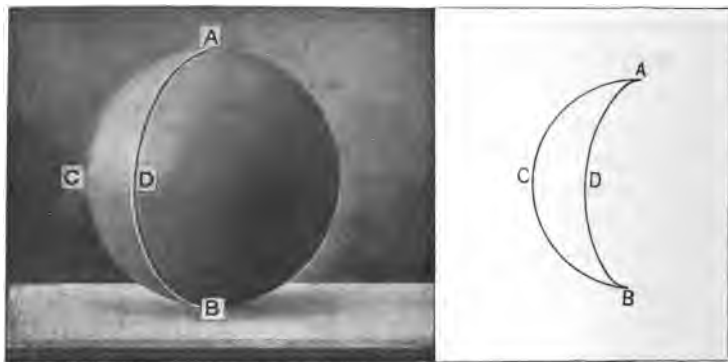
CASE II. *When  $AB$  is a semi-circumference.*

We can show as above that any portion of the shortest line joining  $A$  and  $B$  must be an arc of a great circle, and that therefore the whole must be an arc of a great circle.

Q. E. D.

#### MEASUREMENT OF SPHERICAL FIGURES

**886.** *Def.*—A **lune** is a portion of a spherical surface bounded by two semi-circumferences of great circles; as  $ACBDA$ .

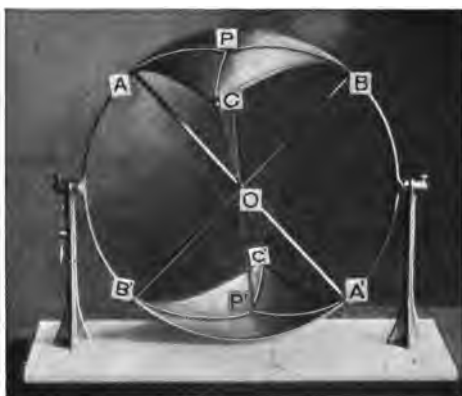


The **angle of a lune** is the angle formed by its bounding arcs.

Thus  $CAD$  is the angle of the lune  $ACBDA$ .

PROPOSITION XXXIII. THEOREM

**887.** *Two symmetrical spherical triangles are equivalent.*



GIVEN two symmetrical triangles  $ABC$  and  $A'B'C'$ .

TO PROVE area  $ABC = \text{area } A'B'C'$ .

Let  $P$  be the pole of the small circle passing through  $A$ ,  $B$ , and  $C$ , and draw the great-circle-arcs  $PA$ ,  $PB$ , and  $PC$ .

Then  $PA = PB = PC$ . § 817

Now place the two triangles vertically opposite to each other and draw the diameter  $POP'$ . § 853

Also draw the great-circle-arcs  $P'A'$ ,  $P'B'$ , and  $P'C'$ .

The vertical triangles  $PBC$  and  $P'B'C'$  are symmetrical and isosceles and therefore equal. § 855

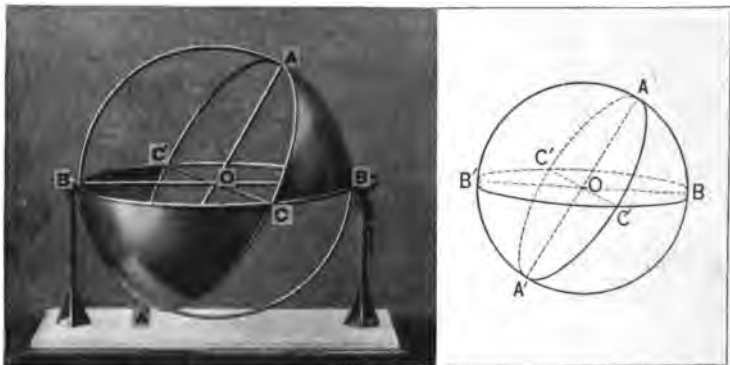
Similarly  $PCA = P'C'A'$  and  $PAB = P'A'B'$ .

That is, the three parts of  $ABC$  are respectively equal to three parts of  $A'B'C'$ .

Therefore area  $ABC = \text{area } A'B'C'$ . Q. E. D.



**888.** COR. I. *If two semi-circumferences of great circles  $BCB'$  and  $ACA'$  intersect on the surface of a hemisphere, the sum of the areas of the two opposite spherical triangles  $ABC$  and  $CA'B'$  is equal to the area of a lune whose angle is equal to  $BCA$ .*

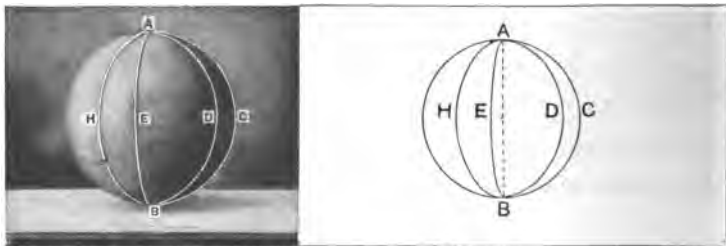


*Hint.*—Area  $ABC$  + area  $CA'B'$  = area  $A'B'C'$  + area  $CA'B'$ .

**889.** COR. II. *Two symmetrical spherical polygons are equivalent.*

PROPOSITION XXXIV. THEOREM

**890.** *Two lunes on the same sphere are equal, if their angles are equal.*



GIVEN—two lunes  $ADBC$  and  $AEBH$  on the same sphere, their angles  $DAC$  and  $HAE$  being equal.

TO PROVE that the lunes are equal.

Since the angles  $DAC$  and  $HAE$  are equal, the plane angles of the dihedral angles  $DABC$  and  $HABE$  are equal.

§ 836 I

Hence these dihedral angles are equal.

§ 572

They can therefore be superposed.

At the same time the lunes coincide.

Therefore the lunes are equal.

Q. E. D.

PROPOSITION XXXV. THEOREM

**891.** *Two lunes on the same sphere are to each other as their angles.*

GIVEN—the lunes  $ADBE$  and  $ACBD$ , whose angles are  $DAE$  and  $CAD$ .

TO PROVE  $\frac{ADBE}{ACBD} = \frac{DAE}{CAD}$ .

CASE I. *When the angles are commensurable (Fig. 1).*

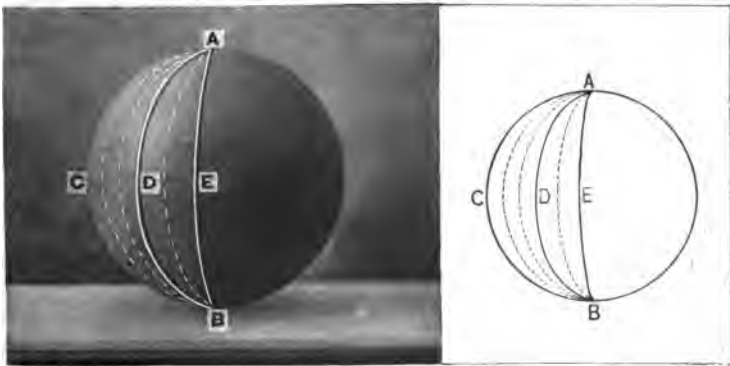


FIG. 1

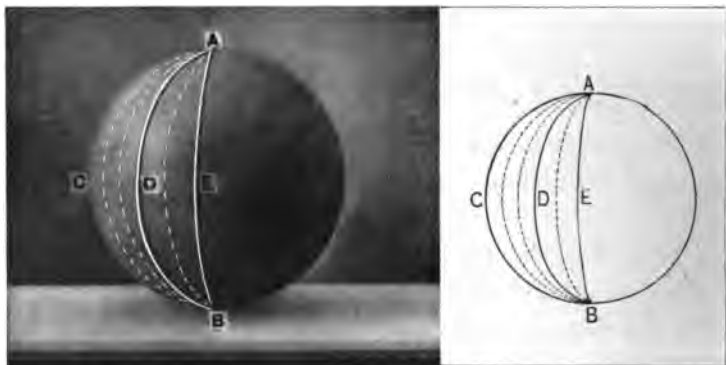


FIG. 1

Suppose a common measure of  $DAE$  and  $CAD$  to be contained twice in  $DAE$  and 3 times in  $CAD$ .

$$\text{Then } \frac{DAE}{CAD} = \frac{2}{3}. \quad \S 180$$

Draw from  $A$  to  $B$  semi-circumferences of great circles dividing the angles  $DAE$  and  $CAD$  into parts each equal to their common measure.

The little lunes thus formed are all equal. § 890

Of these lunes  $ADBE$  contains 2 and  $ACBD$  3.

$$\text{Hence } \frac{ADBE}{ACBD} = \frac{2}{3}. \quad \S 180$$

$$\text{Therefore } \frac{ADBE}{ACBD} = \frac{DAE}{CAD}. \quad \text{Q. E. D.}$$

CASE II. *When the angles are incommensurable* (Fig 2).

Divide  $CAD$  into any number of equal parts by arcs of great circles drawn from  $A$  to  $B$ .

Apply one of these parts to  $DAE$  as many times as it will be contained in it, the final bounding arc taking the position  $AE'B$ .

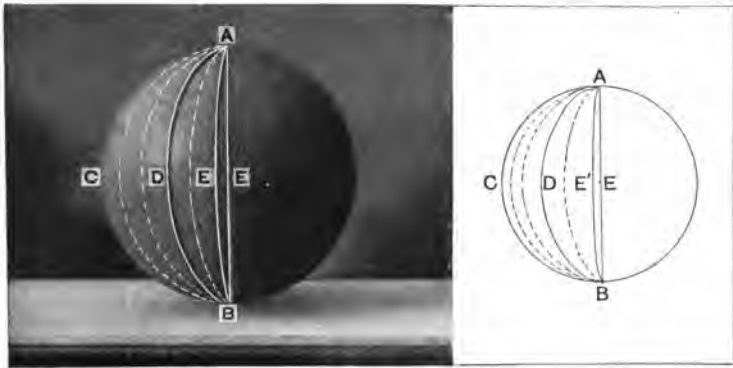


FIG. 2

Since the angles are incommensurable there will be a remainder  $E'AE$  less than one of these parts.

Now the angles  $DAE'$  and  $CAD$  are commensurable.

Therefore 
$$\frac{ADBE'}{ACBD} = \frac{DAE'}{CAD}. \quad \text{Case I}$$

Let the number of parts into which  $CAD$  is divided be indefinitely increased.

Then the angle  $DAE'$  will approach  $DAE$  as a limit.

§ 185

The lune  $ADBE'$  will approach  $ADBE$  as a limit.

Also 
$$\frac{DAE'}{CAD} \text{ will approach } \frac{DAE}{CAD} \text{ as a limit.} \quad \text{§ 190}$$

And 
$$\frac{ADBE'}{ACBD} \text{ will approach } \frac{ADBE}{ACBD} \text{ as a limit.}$$

Therefore 
$$\frac{ADBE}{ACBD} = \frac{DAE}{CAD}. \quad \text{§ 186}$$

Q. E. D.

**892.** COR. I. *A lune is to the surface of the sphere on which it lies as the angle of the lune is to four right angles.*

*Hint.*—The surface of a sphere may be regarded as the limit of a lune whose angle approaches four right angles as a limit.

**893.** COR. II. Let  $A$  denote the angle of a lune measured in the right angle as a unit and  $L$  its surface measured in the tri-rectangular triangle as a unit.

Then the area of the spherical surface will be 8. § 882

$$\text{Hence} \quad \frac{L}{8} = \frac{A}{4}. \quad \S 892$$

$$\text{Therefore} \quad L = 2A.$$

That is, *if the unit angle is the right angle and the unit surface the tri-rectangular triangle, a lune is measured by twice its angle.*

**894.** *Def.*—The **spherical excess** of a spherical triangle is the excess of the sum of its angles over two right angles.

Denoting the angles by  $A, B, C$ , and the spherical excess by  $E$ , we have, taking the right angle as the unit angle,

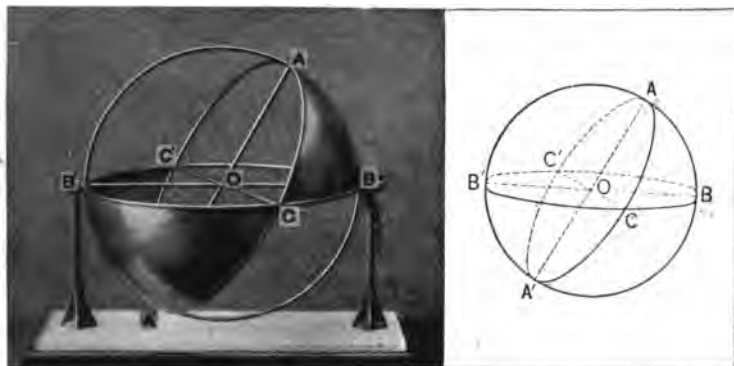
$$E = A + B + C - 2.$$

Thus, if the angles of a spherical triangle are  $45^\circ, 60^\circ, 135^\circ$ , its spherical excess is

$$\left( \frac{45}{90} + \frac{60}{90} + \frac{135}{90} - 2 \right) \text{ right angles} = \frac{2}{3} \text{ right angle.}$$

PROPOSITION XXXVI. THEOREM

**895.** *If the unit angle is the right angle and the unit surface the tri-rectangular triangle, the area of a spherical triangle is measured by its spherical excess.*



**GIVEN** the spherical triangle  $ABC$ .

**TO PROVE** area  $ABC = A + B + C - 2$ ,

the unit angle being the right angle and the unit surface the surface of the tri-rectangular triangle.

Complete the circumference of which  $AB$  is an arc, and let  $BC$  and  $AC$  intersect it again in  $B'$  and  $A'$ .

Then, since  $BCA$  and  $B'CA$  together form a lune whose angle is  $B$ ,

$$\text{area } BCA + \text{area } B'CA = 2B. \quad \S 893$$

Similarly, area  $CAB + \text{area } A'CB = 2A$ .

Also the triangles  $ABC$  and  $CA'B'$  are together equal to a lune whose angle is  $C$ . § 888

Hence area  $ABC + \text{area } CA'B' = 2C$ .

Now the sum of the areas of  $ABC$ ,  $B'CA$ ,  $A'CB$ , and  $CA'B'$  is the area of the surface of a hemisphere, which with the adopted unit is 4.

Hence, adding the three equations above, we have,

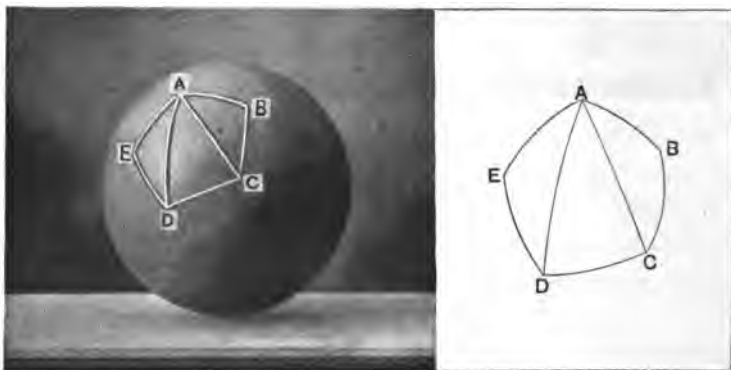
$$2 \text{ area } ABC + 4 = 2A + 2B + 2C.$$

Therefore area  $ABC = A + B + C - 2$ .

**Q. E. D.**

PROPOSITION XXXVII. THEOREM

**896.** *If the unit angle is the right angle, and the unit surface the tri-rectangular triangle, the area of a spherical polygon is measured by the sum of its angles minus twice the number of its sides less two.*



**GIVEN**—the spherical polygon  $ABCDE$ . Denote its area measured in tri-rectangular triangles by  $K$ ; the sum of its angles measured in right angles by  $S$ ; and the number of its sides by  $n$ .

**TO PROVE**  $K = S - 2(n - 2)$ .

Divide the polygon into triangles by diagonals drawn from any vertex  $A$ .

The area of each triangle is measured by the sum of its angles less two. § 895

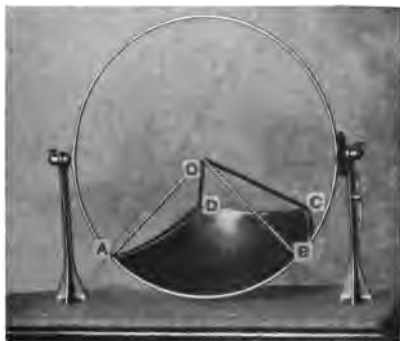
The number of triangles is  $n - 2$ , there being one for every side except the sides intersecting in  $A$ .

Hence the area of the polygon is measured by the sum of the angles of all the triangles minus  $2(n - 2)$ .

But the sum of the angles of all the triangles is equal to the sum of the angles of the polygon.

Therefore  $K = S - 2(n - 2)$ . Q. E. D.

**897. Defs.**—A **spherical pyramid** is a solid bounded by a spherical polygon and the planes of its sides; as  $O-ABCD$ .



The centre of the sphere is called the **vertex** of the spherical pyramid, and the spherical polygon its **base**.

**898. Defs.**—A **spherical ungula**, or **wedge**, is a solid bounded by a lune and the planes of its bounding arcs.

The lune is called the **base** of the ungula; the diameter in which the bounding planes meet is its **edge**.

The angle of the bounding lune is also called the **angle of the ungula**.

**899.** The proofs of the following theorems relating to spherical pyramids and unguas correspond so closely to the proofs of the corresponding theorems relating to spherical polygons and lunes that they are left as exercises for the student.

1. *Two symmetrical triangular spherical pyramids are equivalent.*



- II. *If two great semicircles  $BCB'O$  and  $ACA'O$  (see Fig. § 888) intersect in a hemisphere, the sum of the volumes of the two opposite spherical pyramids  $O - ABC$  and  $O - CA'B'$  is equal to the volume of an ungula whose angle is equal to  $BCA$ .*
- III. *Two symmetrical spherical pyramids are equivalent.*
- IV. *Two unguulas in the same sphere are equal if their angles are equal.*
- V. *Two unguulas in the same sphere are to each other as their angles.*
- VI. *An ungula is to the sphere of which it is a part as its angle is to four right angles.*

If the unit angle is the right angle and the unit solid the tri-rectangular spherical pyramid (that whose base is the tri-rectangular spherical triangle):

- VII. *An ungula is measured by twice its angle.*
- VIII. *The volume of a triangular spherical pyramid is measured by the spherical excess of its base.*
- IX. *The volume of a spherical pyramid is measured by the sum of the angles of its base minus twice the number of its sides less two.*

The following theorems are simple corollaries of the preceding:

- X. *Two triangular spherical pyramids in the same sphere are to each other as their bases.*
- XI. *Any two spherical pyramids in the same sphere are to each other as their bases.*
- XII. *Any spherical pyramid is to the sphere of which it is a part as its base is to the surface of the sphere.*

#### PROBLEMS OF DEMONSTRATION

**900. Exercise.**—The intersection of two spherical surfaces is the circumference of a circle whose plane is perpen-

dicular to the straight line joining the centres of the two spherical surfaces, and whose centre is in that line.

**901. Exercise.**—If from a point without a sphere a tangent and a secant line be drawn, the square of the tangent is equal to the product of the whole secant and its external segment.

**902. Exercise.**—If the centres of three spheres do not lie in the same straight line, their surfaces cannot have more than two points in common. These points lie in a straight line perpendicular to the plane of centres and at equal distances from this plane on opposite sides.

**903. Exercise.**—From a given point on the surface of a sphere, and not on a given great circle, but two great-circle-arcs can be drawn perpendicular to the given great circle; and these are the shortest and longest great-circle-arcs that can be drawn from the point to the given great circle.

**904. Exercise.**—If any number of lines in space meet in a point, the feet of the perpendiculars drawn to these lines from another point lie on the surface of a sphere.

**905. Exercise.**—If from a point within a spherical triangle arcs of great circles are drawn to the extremities of one side, the sum of these arcs is less than the sum of the two other sides of the triangle.

**906. Exercise.**—Any point in the bisector of a spherical angle is equally distant from the sides of the angle.

**907. Exercise.**—The bisectors of the angles of a spherical triangle meet in a point which is equally distant from the sides of the triangle.

**908. Exercise.**—The three medians of a spherical triangle meet in a point.

**909. Exercise.**—The perpendicular bisectors of the sides of a spherical triangle meet in a point.

**910. Exercise.**—If  $a, b, c$  are the sides of a spherical triangle and  $a', b', c'$  the corresponding sides of the polar triangle, if  $a > b > c$ , then  $a' < b' < c'$ .

**911. Exercise.**—Spherical triangles on equal spheres have equal areas if their polar triangles have equal perimeters.

#### LOCI

**912. Exercise.**—Find the locus of a point at a given distance from an indefinite straight line.

**913. Exercise.**—Find the locus of a point at a given distance from a straight line of definite length.

**914. Exercise.**—Find the locus of a point whose distance from a fixed straight line is in a given ratio to its distance from a fixed plane perpendicular to that line.

**915. Exercise.**—Find the locus of a point from which tangent lines drawn to three mutually intersecting spheres are equal.

**916. Exercise.**—Find the locus of the centre of a sphere which is tangent to three given planes.

**917. Exercise.**—Find the locus of a point in space the ratio of whose distances from two given points is constant.

**918. Exercise.**—Find the locus of the centre of the section of a given sphere made by a plane passing through a given point.

**919. Exercise.**—From a fixed point straight lines are drawn to the surface of a sphere. Find the locus of the points which divide these lines in a given ratio.

**920. Exercise.**—Find the locus of a point on the surface of a sphere equidistant from two given points on the surface.

**921. Exercise.**—Find the locus of a point on the surface of a sphere equidistant from three given points on the surface.

**922. Exercise.**—Find the locus of a point in space the ratio of whose distances from two given parallel straight lines is constant.

#### PROBLEMS OF CONSTRUCTION

**923. Exercise.**—Through a given straight line not intersecting a sphere pass a plane tangent to the sphere.

**924. Exercise.**—Construct a spherical surface of given radius:

(a.) Passing through three given points.

(b.) Passing through two given points and tangent to a given plane.

(c.) Passing through two given points and tangent to a given sphere.

(d.) Passing through a given point and tangent to two given planes.

(e.) Passing through a given point and tangent to two given spheres.

(f.) Tangent to three given spheres.

(g.) Tangent to a given plane and two given spheres.

**925. Exercise.**—Bisect a given arc of a great circle.

**926. Exercise.**—Through a given point on a sphere draw a great circle tangent to a given small circle.

#### PROBLEMS FOR COMPUTATION

**927. (1.)** The radius of a sphere is 25 in. Find the area of a section made by a plane 10 in. distant from its centre.

(2.) What is the radius of a sphere inscribed in a regular tetrahedron whose total area is 4 sq. m.?

(3.) What is the radius of a spherical surface passing through four points each of which is 9 cm. distant from the other three?

(4.) If the volume of a sphere is 12 cu. m., what is the volume of a spherical wedge the angles of whose base are  $40^\circ$ ?

(5.) If the area of a spherical surface is 100 sq. ft., what is the area of a spherical triangle whose angles are  $30^\circ$ ,  $120^\circ$ , and  $150^\circ$ ?

(6.) The volume of a sphere is 1000 cu. in. What is the volume of a spherical pyramid the angles of whose base are  $30^\circ$ ,  $90^\circ$ ,  $130^\circ$ , and  $160^\circ$ ?

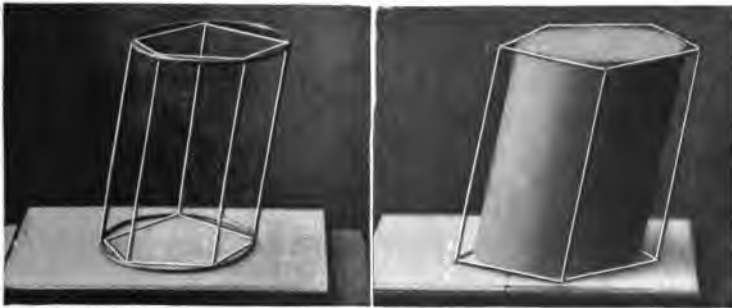
# GEOMETRY OF SPACE

## BOOK IX

### MEASUREMENT OF THE CYLINDER, CONE, AND SPHERE

#### THE CYLINDER

**928.** *Def.*—A prism is **inscribed in a cylinder** when its lateral edges are elements of the cylinder and its bases are in the planes of the bases of the cylinder.



**929.** *Def.*—A prism is **circumscribed about a cylinder** when its lateral faces are tangent to the cylinder and its bases are in the planes of the bases of the cylinder.

**930.** *Def.*—A right section of a cylinder is a section made by a plane perpendicular to its elements.



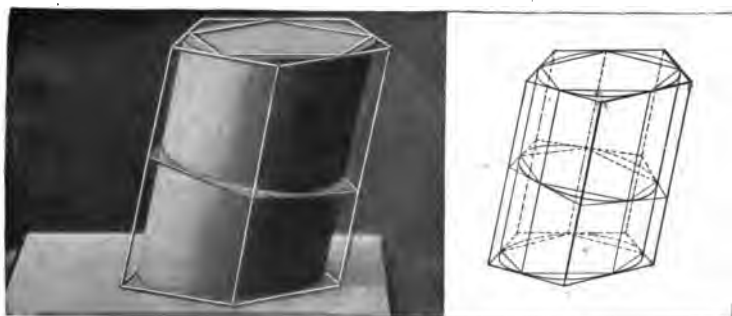
**931.** *Remark.*—From the preceding definitions it follows immediately that the bases of an inscribed prism are inscribed in the bases of the cylinder; the bases of a circumscribed prism are circumscribed about the bases of the cylinder; and that a plane forming a right section of a cylinder forms a right section of every inscribed and every circumscribed prism.

**932.** *Def.*—The lateral area of a cylinder is the area of its lateral surface.

#### PROPOSITION I. THEOREM

**933.** *If the number of lateral faces of a prism inscribed in or circumscribed about a cylinder be indefinitely increased so that each one becomes indefinitely small, then*

- I. *Any right section of the prism approaches a right section of the cylinder as a limit.*
- II. *The lateral area of the prism approaches the lateral area of the cylinder as a limit.*
- III. *The volume of the prism approaches the volume of the cylinder as a limit.*



*Proof.*—I. A plane which forms a right section of the prism will also form a right section of the cylinder. § 931

When the number of lateral faces of the prism is indefinitely increased so that each one becomes indefinitely small, the number of sides of the right section will be indefinitely increased, and each will become indefinitely small.

Therefore the right section of the prism approaches the right section of the cylinder as a limit. § 490

Q. E. D.

II. The lateral surface of the prism can be generated by a straight line moving about its right section as a directrix, provided this line remains parallel to the lateral edges and is terminated by the two bases. § 632

As the number of lateral faces increases indefinitely, the directrix of this line approaches the right section of the cylinder as a limit.

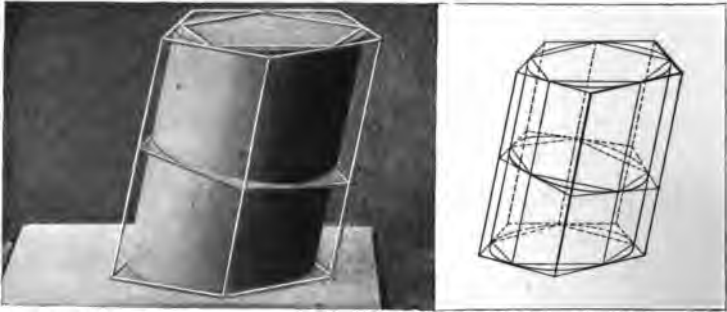
Hence the limit of the surface generated by this line is the surface generated by it when the directrix is the perimeter of the right section of the cylinder.

But this surface is the lateral surface of the cylinder. § 766

Therefore the limit of the lateral area of the prism is the lateral area of the cylinder.

Q. E. D.





III. Let  $B'$ ,  $B''$  be the respective bases of a circumscribed and corresponding inscribed prism,  $V'$ ,  $V''$  their respective volumes, and  $H$  their common altitude.

Then  $V' = B' \times H$ , and  $V'' = B'' \times H$ . § 676

Hence  $V' - V'' = (B' - B'') \times H$ .

Now by increasing indefinitely the number of lateral faces of the prisms, and consequently the number of sides of their bases, the difference  $B' - B''$  can be made as small as we please. § 490

Hence  $(B' - B'') \times H$  can be made as small as we please.

§ 187

Hence its equal  $V' - V''$  can be made as small as we please.

But the volume of the cylinder is always intermediate between  $V'$  and  $V''$ . Ax. 10

Therefore the difference between the volume of the cylinder and either  $V'$  or  $V''$  can be made as small as we please.

But  $V'$  and  $V''$  can never equal the volume of the cylinder. Ax. 10

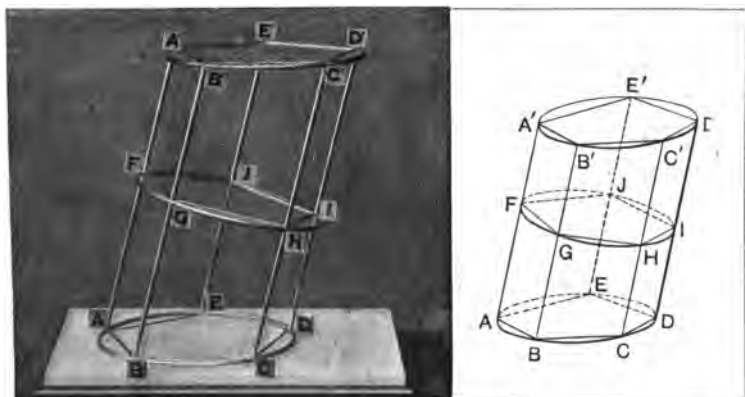
Therefore the volume of the cylinder is the common limit of  $V'$  and  $V''$ .

§ 185

Q. E. D

PROPOSITION II. THEOREM

934. *The lateral area of a cylinder is equal to the product of the perimeter of a right section and an element.*



GIVEN—the cylinder  $AD'$ , of which  $P$  is the perimeter of the right section  $FGHIJ$ ,  $E$  an element, and  $S$  the lateral area.

TO PROVE  $S = P \times E$ .

Inscribe in the cylinder a prism. Let  $P'$  be the perimeter of its right section and  $S'$  its lateral area.

Its lateral edge is equal to  $E$ . § 545

Hence  $S' = P' \times E$ . § 649

Now let the number of lateral faces of the prism be indefinitely increased.

Then  $S'$  approaches  $S$  as a limit, § 933 II

$P'$  approaches  $P$  as a limit, § 933 I

and  $P' \times E$  approaches  $P \times E$  as a limit. § 189

Therefore  $S = P \times E$ . § 186

Q. E. D.

a better proof:  
 The area of the lateral surface of the prism is  $P' \times E$ .  
 The area of the lateral surface of the cylinder is  $S$ .  
 The area of the lateral surface of the cylinder is also a limit of  $P' \times E$  as the number of lateral faces of the prism is indefinitely increased.  
 surface =  $P \times E$

**935. Def.**—The **altitude** of a cylinder is the perpendicular distance between its bases.

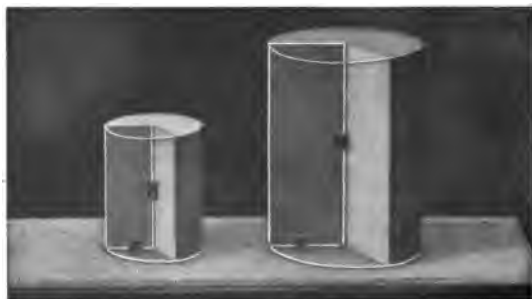
**936. COR. I.** *The lateral area of a right cylinder is equal to the product of the perimeter of its base by its altitude.*

**937. COR. II.** Let  $H$  denote the altitude,  $R$  the radius,  $S$  the lateral area, and  $T$  the total area of a cylinder of revolution.



Then  $S = 2\pi RH$ ,  
and  $T = 2\pi RH + 2\pi R^2 = 2\pi R(H + R)$ .

**938. Def.**—**Similar cylinders of revolution** are cylinders formed by the revolution of similar rectangles about homologous sides.



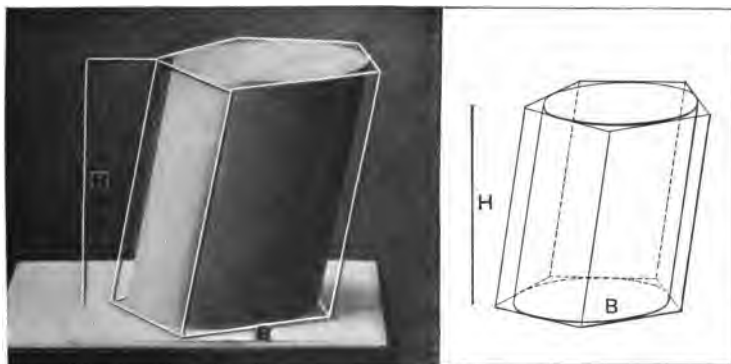
**939.** COR. III. *The lateral areas, or the total areas, of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of their radii.*

OUTLINE PROOF:  $\frac{S}{s} = \frac{2\pi RH}{2\pi rh} = \frac{R}{r} \times \frac{H}{h} = \frac{H}{h} \times \frac{H}{h} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$ .

$$\frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} = \frac{R}{r} \times \frac{H+R}{h+r} = \frac{H}{h} \times \frac{H}{h} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$$

PROPOSITION III. THEOREM

**940.** *The volume of a cylinder is equal to the product of its base and altitude.*



GIVEN—a cylinder, of which  $B$  is the base,  $H$  the altitude, and  $V$  the volume.

TO PROVE  $V = B \times H$ .

Circumscribe about the cylinder a prism. Denote its base by  $B'$  and its volume by  $V'$ .

Its altitude is  $H$ . § 545

Hence  $V' = B' \times H$ . § 676

Now let the number of lateral faces of the prism be indefinitely increased.

Then	$V'$ approaches $V$ as a limit,	§ 933 III
	$B'$ approaches $B$ as a limit,	§ 490
and	$B' \times H$ approaches $B \times H$ as a limit.	§ 189
Therefore	$V = B \times H.$	§ 186
		Q. E. D.

**941. COR. I.** Let  $H$  be the altitude,  $R$  the radius, and  $V$  the volume of a circular cylinder.

Then  $V = \pi R^2 H.$

**942. COR. II.** *The volumes of two similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.*

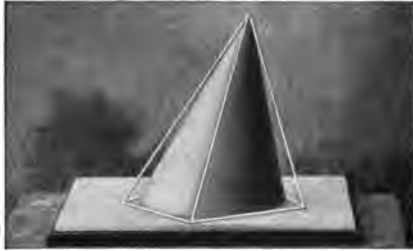
OUTLINE PROOF: 
$$\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} = \frac{R^2}{r^2} \times \frac{H}{h} = \frac{H^2}{h^2} \times \frac{H}{h} = \frac{H^3}{h^3} = \frac{R^3}{r^3}.$$

#### THE CONE

**943. Def.**—A pyramid is inscribed in a cone when its lateral edges are elements of the cone and its base is in the plane of the base of the cone.



**944. Def.**—A pyramid is circumscribed about a cone when its lateral faces are tangent to the cone and its base is in the plane of the base of the cone.



**945. Remark.**—From these definitions it follows immediately that the base of an inscribed pyramid is inscribed in the base of the cone and that the base of a circumscribed pyramid is circumscribed about the base of the cone.

**946. Defs.**—A **truncated cone** is the portion of a cone contained between its base and a plane cutting all its elements.

The base of the cone and the section made by the cutting plane are called the **bases** of the truncated cone.

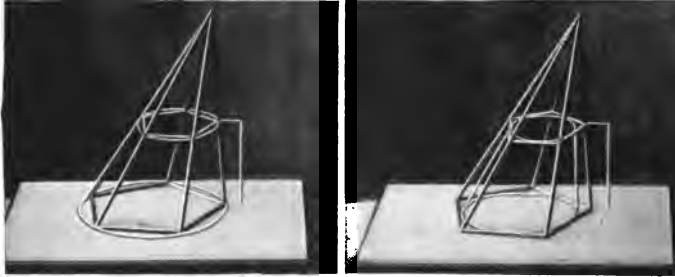


FRUSTUM

TRUNCATED CONE

**947. Def.**—A **frustum of a cone** is a truncated cone whose bases are parallel.

**948. Def.**—If a pyramid is inscribed in or circumscribed about a cone, a plane which cuts from the cone a truncated cone cuts from the pyramid a truncated pyramid, which may be said to be **inscribed** in or **circumscribed** about the truncated cone.



**949. Def.**—The **lateral area** of a cone is the area of its lateral surface.

PROPOSITION IV: THEOREM

**950.** *If the number of lateral faces of a pyramid inscribed in or circumscribed about a cone be indefinitely increased so that each one becomes indefinitely small, then*

- I. *Any section of the pyramid approaches the section of the cone by the same plane as a limit.*
- II. *The lateral area of the pyramid approaches the lateral area of the cone as a limit.*
- III. *The volume of the pyramid approaches the volume of the cone as a limit.*

The proof of this proposition is analogous to that of Proposition I., and is therefore left to the student.

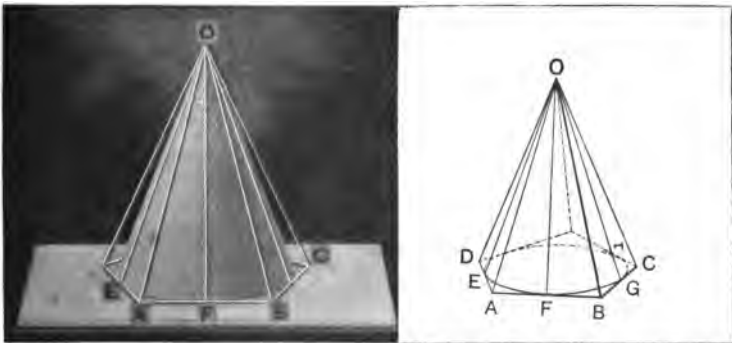
**951. Remark.**—The proposition obtained from the preceding by substituting the words “frustum of a pyramid” and “frustum of a cone” for “pyramid” and “cone” can be proved in the same way.

**952. Def.**—Any element of a cone of revolution is called its **slant height**.

**953. Exercise.**—Prove that the slant height of a regular pyramid circumscribed about a cone of revolution is equal to the slant height of the cone of revolution.

PROPOSITION V. THEOREM

**954.** *The lateral area of a cone of revolution is equal to one-half the product of the circumference of its base by its slant height.*



**GIVEN**—the cone of revolution  $O-EFGH$ . Denote its slant height  $OE$  by  $L$ , the circumference of its base by  $C$ , and its lateral area by  $S$ .

**TO PROVE**  $S = \frac{1}{2} C \times L$ .

Circumscribe about the cone a regular pyramid. Denote the perimeter of its base by  $C'$  and its lateral area by  $S'$ .

Its slant height will also be  $L$ . § 953

Hence  $S' = \frac{1}{2} C' \times L$ . § 688



Now let the number of lateral faces of the regular pyramid be indefinitely increased.

Then  $S'$  approaches  $S$  as a limit. § 950 II

And  $C'$  approaches  $C$  as a limit. § 490

Hence  $\frac{1}{2}C' \times L$  approaches  $\frac{1}{2}C \times L$  as a limit. § 189

Therefore  $S = \frac{1}{2}C \times L$ . § 186

Q. E. D.

**955.** COR. I. Let  $R$  denote the radius,  $L$  the slant height,  $S$  the lateral area, and  $T$  the total area of a cone of revolution.

Then  $S = \frac{1}{2}2\pi R \times L = \pi RL$ .

And  $T = \pi RL + \pi R^2 = \pi R(L + R)$ .

**956.** COR. II. The formula for the lateral area may be written

$$S = 2\pi \frac{R}{2} \times L.$$

Now, if  $K$  is the radius of a section half-way between the vertex and base,

$$K = \frac{1}{2}R.$$

Therefore  $S = 2\pi K \times L$ .



That is, *the lateral area of a cone of revolution is equal to the circumference of a section half-way between its vertex and base multiplied by its slant height.*

**957.** Def.—The **altitude** of a cone is the perpendicular distance from its vertex to its base.

**958. Def.**—Similar cones of revolution are cones formed by the revolution of similar right triangles about homologous sides.

**959. COR. III.** *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases.*



$\pi RL$   $R \times L$   
 $\pi r l$   $r \times l$   
 $R^2$   $l^2$

*Hint.*—The method of proof is the same as that followed in § 939.

**960. Def.**—The portion of an element of a cone of revolution included between the bases of a frustum is called the **slant height of the frustum**.



**961. Exercise.**—Prove that the slant height of a frustum of a regular pyramid which is circumscribed about a frustum of a cone of revolution is equal to the slant height of the frustum of a cone.

## PROPOSITION VI. THEOREM

**962.** *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of its bases multiplied by its slant height.*



**GIVEN** a frustum of a cone of revolution.

Denote the circumferences of its bases by  $C$  and  $c$ , its slant height by  $L$ , and its lateral area by  $S$ .

**TO PROVE**  $S = \frac{1}{2}(C + c) \times L$ .

Circumscribe about the frustum a frustum of a regular pyramid.

Denote the perimeters of its bases by  $C'$  and  $c'$ , and its lateral area by  $S'$ . Its slant height will also be  $L$ . § 961

Hence  $S' = \frac{1}{2}(C' + c') \times L$ . § 693

Now let the number of lateral faces of the frustum of a regular pyramid be indefinitely increased.

Then  $S'$  approaches  $S$  as a limit, § 951

$C' + c'$  approaches  $C + c$  as a limit, § 490

and  $\frac{1}{2}(C' + c') \times L$  approaches  $\frac{1}{2}(C + c) \times L$  as a limit. § 189

Therefore  $S = \frac{1}{2}(C + c) \times L$ . § 186

Q. E. D.

**963.** COR. I. If  $R$  and  $r$  are the radii of the bases,  $L$  the slant height, and  $S$  the lateral area of a frustum of a cone of revolution,

$$S = \frac{1}{2}(2\pi R + 2\pi r) \times L = \pi(R + r) \times L.$$

**964.** COR. II. The last formula may be written

$$S = 2\pi \frac{R+r}{2} \times L.$$

If  $K$  is the radius of a section half-way between the bases of the frustum,

$$K = \frac{R+r}{2}.$$

Hence

$$S = 2\pi K \times L.$$



That is, *the lateral area of a frustum of a cone of revolution is equal to the circumference of a section half-way between its bases multiplied by its slant height.*

#### PROPOSITION VII. THEOREM

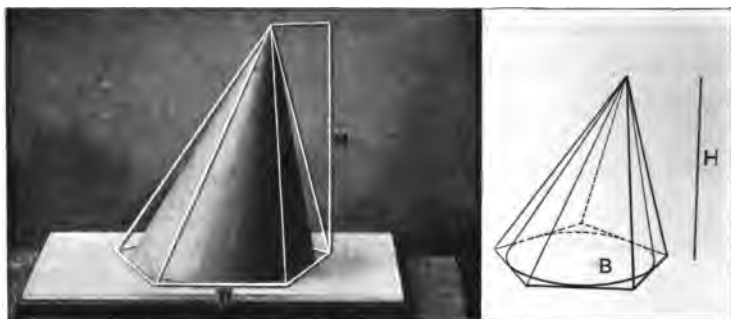
**965.** *The volume of a cone is equal to one-third the product of its base and altitude.*

GIVEN—any cone, of which  $B$  is the base,  $H$  the altitude, and  $V$  the volume.

TO PROVE

$$V = \frac{1}{3} B \times H.$$

Circumscribe about the cone a pyramid. Denote its base by  $B'$ , and its volume by  $V'$ . Its altitude is  $H$ .



Then  $V' = \frac{1}{3}B' \times H.$  § 704

Now let the number of lateral faces of the pyramid be indefinitely increased.

Then  $V'$  approaches  $V$  as a limit, § 950 III

$B'$  approaches  $B$  as a limit, § 490

and  $\frac{1}{3}B' \times H$  approaches  $\frac{1}{3}B \times H$  as a limit. § 189

Therefore  $V = \frac{1}{3}B \times H.$  Q. E. D.

**966.** COR. I. If the base of the cone is a circle of radius  $R$ ,

$$V = \frac{1}{3}\pi R^2 H.$$

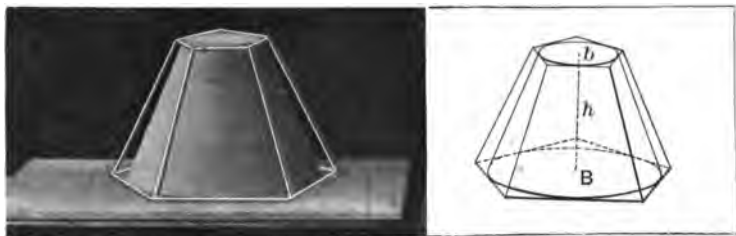
**967.** COR. II. *The volumes of two similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

*Hint.*—The method of proof is the same as that followed in § 942.

**968.** *Def.*—The **altitude** of a frustum of a cone is the perpendicular distance between its bases.

#### PROPOSITION VIII. THEOREM

**969.** *A frustum of a cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



GIVEN—a frustum of a cone. Denote its bases by  $B$  and  $b$ , its altitude by  $h$ , and its volume by  $V$ .

TO PROVE— $V = \frac{1}{3}h(B + b + \sqrt{B \times b})$ , which is the algebraic statement of the theorem.

Circumscribe about the frustum of a cone a frustum of a pyramid. Denote its bases by  $B'$  and  $b'$ , and its volume by  $V'$ .

Its altitude will be  $h$ . § 565

Hence  $V' = \frac{1}{3}h(B' + b' + \sqrt{B' \times b'})$ . § 713

Now let the number of lateral faces of the frustum of a pyramid be indefinitely increased.

Then  $V'$  approaches  $V$  as a limit, § 951

$B'$  approaches  $B$  as a limit, § 490

$b'$  approaches  $b$  as a limit,

and  $\frac{1}{3}h(B' + b' + \sqrt{B' \times b'})$  approaches  $\frac{1}{3}h(B + b + \sqrt{B \times b})$ .

Therefore  $V = \frac{1}{3}h(B + b + \sqrt{B \times b})$ . § 186

Q. E. D.

**970. COR.** If the frustum is the frustum of a circular cone, let  $R$  and  $r$  be the radii of its bases.

Then  $B = \pi R^2$ ,  $b = \pi r^2$ ,  $\sqrt{B \times b} = \pi Rr$ .

Therefore  $V = \frac{1}{3}\pi h(R^2 + r^2 + Rr)$ .

## THE SPHERE

**971. Defs.**—A **zone** is a portion of the surface of a sphere bounded by the circumferences of two circles whose planes are parallel.



The bounding circumferences are called the **bases**, and the perpendicular distance between their planes the **altitude** of the zone.

**972. Def.**—If the plane of one bounding circumference is tangent to the sphere, the zone is called a **zone of one base**.

**973. Defs.**—A **spherical segment** is a portion of a sphere contained between two parallel planes.

The bounding circles are called the **bases**, and the perpendicular distance between their planes the **altitude** of the segment.

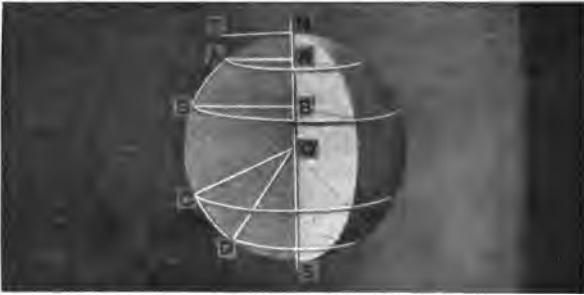
**974. Def.**—A **spherical segment of one base** is a spherical segment one of whose bounding planes is tangent to the sphere.

The curved surface of a spherical segment is a zone.

**975. Defs.**—If a semicircle is revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a **spherical sector**.

The zone generated by the base of the sector of the semicircle is called the **base** of the spherical sector.

**976. Remarks.**—Suppose a sphere generated by the revolution of the semicircle  $HAS$  about its diameter  $HS$  as an axis. Let  $AA'$  and  $BB'$  be two lines perpendicular to  $HS$ , and let  $OC$  and  $OD$  be radii of the semicircle.



Then the arc  $AB$  generates a zone whose altitude is  $A'B'$ ; the points  $A$  and  $B$  generate the bases of the zone.

The arc  $HA$  generates a zone of one base.

The figure  $AA'B'B$  generates a spherical segment whose altitude is  $A'B'$ ; the lines  $AA'$  and  $BB'$  generate the bases of the spherical segment.

The figure  $HAA'$  generates a spherical segment of one base.

The sector  $COD$  of the semicircle generates a spherical sector. This spherical sector is bounded by three curved surfaces, namely: the two conical surfaces generated by the radii  $OC$  and  $OD$ , and the zone generated by the arc  $CD$ .



## PROPOSITION IX. LEMMA

**977.** *The area of the surface generated by a straight line revolving about an axis in its plane (not crossing the straight line) is equal to the projection of the line on the axis multiplied by the circumference of the circle whose radius is the perpendicular to the line drawn at its middle point and terminated in the axis.*



FIG. 1

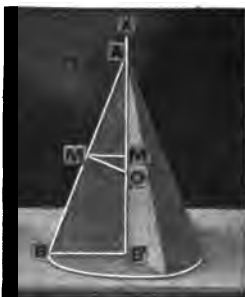


FIG. 2



FIG. 3

**GIVEN**—the straight lines  $AB$  and  $XY$  in the same plane,  $XY$  not crossing  $AB$ . Let  $S$  denote the area of the surface generated by revolving  $AB$  about  $XY$  as an axis.

Draw a perpendicular  $MO$  to  $AB$  at its middle point  $M$  cutting  $XY$  in  $O$ , and let  $A'B'$  be the projection of  $AB$  on  $XY$ .

**TO PROVE**  $S = A'B' \times 2\pi MO$ .

**CASE I.** *When  $AB$  is parallel to  $XY$  (Fig. 1).*

The surface generated in this case is the lateral surface of a cylinder of revolution. § 778

Hence  $S = AB \times 2\pi BB'$ . § 937

Or  $S = A'B' \times 2\pi MO$ . § 117

**Q. E. D.**

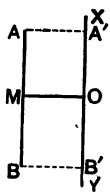


FIG. 1

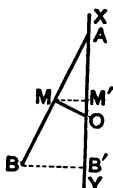


FIG. 2

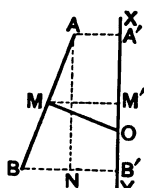


FIG. 3

CASE II. When one end  $A$  of  $AB$  is in  $XY$  (Fig. 2).

The surface generated in this case is the lateral surface of a cone of revolution. § 794

Draw  $MM'$  perpendicular to  $XY$ .

Then  $S = AB \times 2\pi MM'$ . § 956

The triangles  $AB'B$  and  $MM'O$  are similar. § 286

Hence  $\frac{AB'}{AB} = \frac{MM'}{MO} = \frac{2\pi MM'}{2\pi MO}$ . § 274

Hence  $AB \times 2\pi MM' = AB' \times 2\pi MO$ . § 250

Therefore  $S = AB' \times 2\pi MO$ . Q. E. D.

CASE III. When  $AB$  is not parallel to  $XY$  and does not meet  $XY$  (Fig. 3).

The surface generated in this case will be that of a frustum of a cone of revolution.

Draw  $MM'$  perpendicular to  $XY$  and  $AN$  perpendicular to  $BB'$ .

Then  $S = AB \times 2\pi MM'$ . § 964

The triangles  $ANB$  and  $MM'O$  are similar. § 286

Hence  $\frac{AN}{AB} = \frac{MM'}{MO} = \frac{2\pi MM'}{2\pi MO}$ .

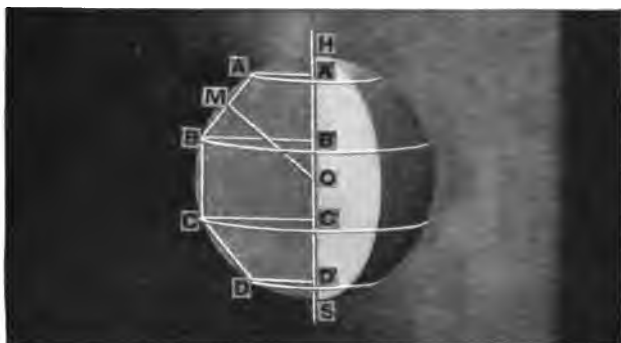
Hence  $AB \times 2\pi MM' = AN \times 2\pi MO = A'B' \times 2\pi MO$ .

Therefore  $S = A'B' \times 2\pi MO$ . Q. E. D.

**978. Def.**—A broken line is a line which is not straight, but consists of several straight parts.

## PROPOSITION X. THEOREM

**979.** *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*



**GIVEN**—a zone formed by the revolution of the arc  $AD$  of the semicircle  $HAS$  about its diameter  $HS$  as an axis. Let  $A'D'$  be the altitude of the zone and  $O$  the centre of the semicircle.

**TO PROVE**                      area zone  $AD = A'D' \times 2\pi OA$ .

Divide the arc  $AD$  into any number of equal parts,  $AB$ ,  $BC$ ,  $CD$ . Draw the chords  $AB$ ,  $BC$ ,  $CD$ .

Also draw  $AA'$ ,  $BB'$ ,  $CC'$ ,  $DD'$  perpendicular to  $HS$  and  $OM$  perpendicular to  $AB$ .

Denote by "area  $AB$ " the area of the surface generated by the straight line  $AB$  in revolving about  $HS$ .

Then                      area  $AB = A'B' \times 2\pi OM$ .                      § 977

Similarly              area  $BC = B'C' \times 2\pi OM$ ,                      §§ 164, 170

and                      area  $CD = C'D' \times 2\pi OM$ .

Adding these equations we have

$$\begin{aligned} \text{area broken line } ABCD &= (A'B' + B'C' + C'D') \times 2\pi OM \\ &= A'D' \times 2\pi OM. \end{aligned}$$

Now let the number of divisions of the arc  $AD$  be increased indefinitely.

Then the broken line approaches the arc  $AD$  as a limit and  $OM$  approaches the radius  $OA$  of the sphere as a limit.

Moreover, the limit of the surface generated by the broken line  $ABCD$  will be the surface generated by the limit of the broken line, that is, by the arc  $AD$ .

This latter is the zone  $AD$ .

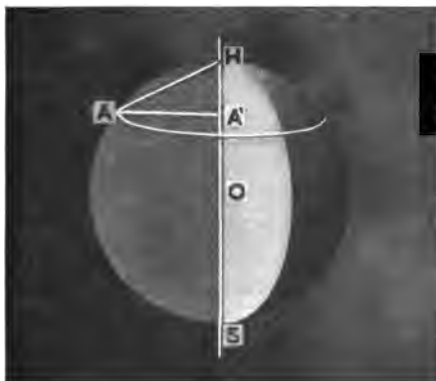
Therefore area zone  $AD = A'D' \times 2\pi OA$ . § 186  
Q. E. D.

**980.** COR. I. Let  $S$  denote the area of the zone,  $H$  its altitude, and  $R$  the radius of the sphere.

Then  $S = 2\pi RH$ .

**981.** COR. II. *Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.*

**982.** COR. III. *A zone of one base is equivalent to a circle whose radius is the chord of the generating arc of the zone.*



OUTLINE PROOF: Area zone  $HA = 2\pi OH \times HA' = \pi HS \times HA' = \pi \overline{HA'}$

**983. COR. IV.** *The surface of a sphere is equivalent to four great circles.*

*Hint.*—The surface may be considered to be a zone whose altitude is the diameter of the sphere.

Hence its area is  $2\pi R \times 2R = 4\pi R^2$ .

**984. COR. V.** *Two spherical surfaces are to each other as the squares of their radii or as the squares of their diameters.*

PROPOSITION XI. LEMMA

**985.** *If a triangle revolve about an axis situated in its plane and passing through the vertex without crossing its surface, the volume generated will be equal to the area generated by the base multiplied by one-third of the altitude.*

GIVEN — the triangle  $ABC$  revolving about an axis  $XY$  passing through the vertex  $A$  without crossing the triangle. Let the altitude of the triangle be  $AD$ .

TO PROVE      vol. gen. by  $ABC = \text{area } BC \times \frac{1}{3}AD$ .

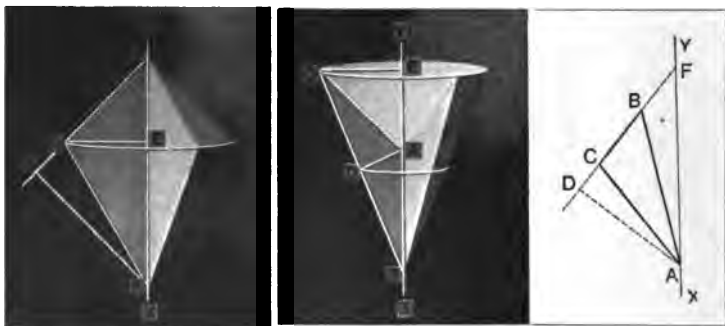


FIG. 1

FIG. 2

FIG. 3

CASE I. *When one side of the triangle  $ABC$ , as  $AB$ , lies in the axis.*

Draw  $CE$  perpendicular to the axis.

If this perpendicular falls within the triangle (Fig. 1), the volume generated by the triangle  $ABC$  is the sum of the volumes generated by the triangles  $BEC$  and  $AEC$ . That is,

$$\text{vol. } ABC = \text{vol. } BEC + \text{vol. } AEC. \quad (1)$$

If the perpendicular falls without the triangle (Fig. 2), the volume generated by the triangle  $ABC$  is the difference of the volumes generated by the triangles  $BEC$  and  $AEC$ . That is,

$$\text{vol. } ABC = \text{vol. } BEC - \text{vol. } AEC. \quad (2)$$

Now in either case

$$\text{vol. } BEC = \frac{1}{3}\pi \overline{EC}^2 \times BE \quad \S 966$$

and 
$$\text{vol. } AEC = \frac{1}{3}\pi \overline{EC}^2 \times AE.$$

Substituting these values in (1), we have

$$\text{vol. } ABC = \frac{1}{3}\pi \overline{EC}^2 \times (BE + AE).$$

For this case  $BE + AE = AB$ .

Substituting in (2), we have

$$\text{vol. } ABC = \frac{1}{3}\pi \overline{EC}^2 \times (BE - AE).$$

For this case  $BE - AE = AB$ .

Hence, in either case,

$$\begin{aligned} \text{vol. } ABC &= \frac{1}{3}\pi \overline{EC}^2 \times AB \\ &= \frac{1}{3}\pi EC \times EC \times AB. \end{aligned}$$

But  $EC \times AB = BC \times AD$ ,

since each side is twice the area of the triangle  $ABC$ .

Therefore  $\text{vol. } ABC = \frac{1}{3}\pi EC \times BC \times AD$ .

But  $\pi EC \times BC$  is the area of the conical surface generated by  $BC$ . § 955

Therefore  $\text{vol. } ABC = \text{area } BC \times \frac{1}{3}AD$ . Q. E. D.

CASE II. *When the triangle  $ABC$  has neither side coinciding with the axis, and the base  $BC$  when produced meets the axis in  $F$  (Fig. 3).*

Then  $\text{vol. } ABC = \text{vol. } AFC - \text{vol. } AFB$ .

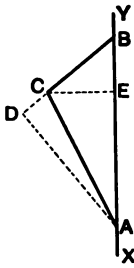


FIG. 1

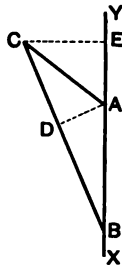


FIG. 2

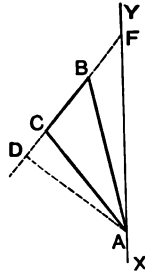


FIG. 3

But  $\text{vol. } AFC = \text{area } FC \times \frac{1}{3}AD,$   
 and  $\text{vol. } AFB = \text{area } FB \times \frac{1}{3}AD.$  Case I  
 Therefore  $\text{vol. } ABC = (\text{area } FC - \text{area } FB) \times \frac{1}{3}AD$   
 $= \text{area } BC \times \frac{1}{3}AD.$  Q. E. D

CASE III. *When the base BC of the triangle ABC is parallel to the axis.*

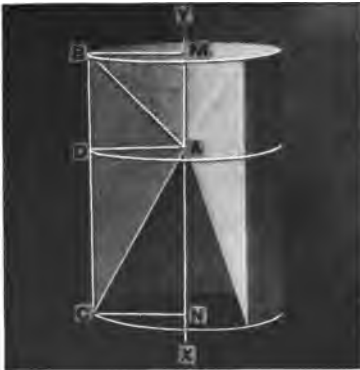


FIG. 4

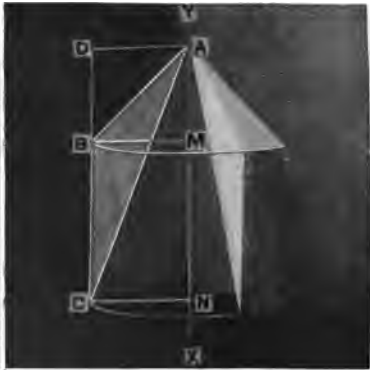


FIG. 5

According as  $AD$  falls within (Fig. 4) or without (Fig. 5) the triangle, we have

$$\text{vol. } ABC = \text{vol. } ADC + \text{vol. } ADB, \quad (3)$$

or  $\text{vol. } ABC = \text{vol. } ADC - \text{vol. } ADB. \quad (4)$

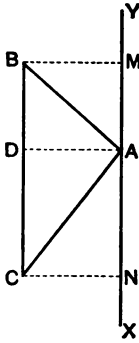


FIG. 4

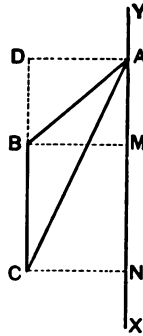


FIG. 5

Draw  $BM$  and  $CN$  perpendicular to  $XY$ .

Now for either figure

$$\begin{aligned} \text{vol. } ADC &= \text{vol. } ADCN - \text{vol. } ACN \\ &= \pi \overline{NC}^2 \times AN - \frac{1}{3} \pi \overline{NC}^2 \times AN \quad \text{\S\S 941, 966} \\ &= \frac{2}{3} \pi \overline{NC}^2 \times AN = \frac{2}{3} \pi AD^2 \times DC \\ &= 2\pi AD \times DC \times \frac{1}{3} AD. \end{aligned}$$

But  $2\pi AD \times DC$  is the area of the cylindrical surface generated by  $DC$ . § 937

Therefore  $\text{vol. } ADC = \text{area } DC \times \frac{1}{3} AD$ .

Similarly  $\text{vol. } ADB = \text{area } DB \times \frac{1}{3} AD$ .

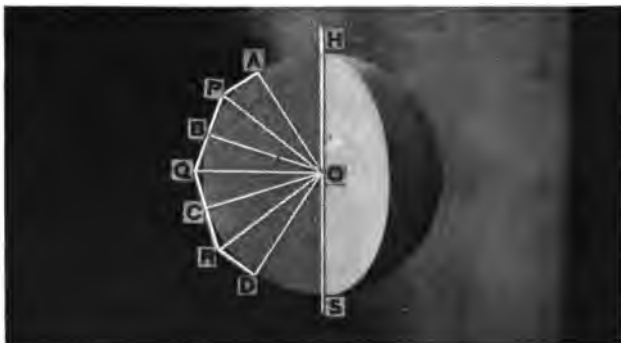
Now, substituting these values for  $\text{vol. } ADC$  and  $\text{vol. } ADB$  in equations (3) and (4), and remembering that equation (3) applies to Fig. 4 and equation (4) to Fig. 5, we get

$$\text{vol. } ABC = \text{area } BC \times \frac{1}{3} AD. \quad \text{Q. E. D.}$$

PROPOSITION XII. THEOREM

**986.** *The volume of a spherical sector is equal to the area of the zone which forms its base multiplied by one-third the radius of the sphere.*





GIVEN—a spherical sector, formed by the revolution of the sector  $AOD$  of the semicircle  $HAS$  about its diameter  $HS$  as an axis.

TO PROVE—vol. sph. sect.  $AOD = \text{area zone } AD \times \frac{1}{3}OA$ .

Divide the arc  $AD$  into any number of equal parts,  $AB$ ,  $BC$ ,  $CD$ .

At  $A$ ,  $B$ ,  $C$ , and  $D$  draw tangents  $AP$ ,  $PQ$ ,  $QR$ ,  $RD$ . Draw  $OB$ ,  $OC$ ,  $OP$ ,  $OQ$ ,  $OR$ .

The volume generated by the polygon  $OAPQRD$  is the sum of the volumes generated by the triangles  $OAP$ ,  $OPQ$ ,  $OQR$ ,  $ORD$ .

$$\begin{aligned} \text{But} \quad \text{vol. } AOP &= \text{area } AP \times \frac{1}{3}OA && \S 985 \\ \text{vol. } OPQ &= \text{area } PQ \times \frac{1}{3}OB = \text{area } PQ \times \frac{1}{3}OA \\ &\text{etc.} \end{aligned}$$

Hence

$$\begin{aligned} \text{vol. } OAPQRD &= (\text{area } AP + \text{area } PQ + \text{etc.}) \times \frac{1}{3}OA \\ &= \text{area } APQRD \times \frac{1}{3}OA. \end{aligned}$$

Now let the number of divisions of the arc  $AD$  be indefinitely increased.

Then broken line  $APQRD$  approaches arc  $AD$  as a limit ;  
 surface generated by the broken line approaches surface generated by the arc as a limit ;  
 that is, surface generated by the broken line approaches the zone  $AD$  as a limit ;  
 volume generated by the polygon approaches volume generated by the sector ;

that is, volume generated by the polygon approaches volume spherical sector  $AOD$ ;  
and  $OA$  is constant.

Therefore vol. sph. sect.  $AOD = \text{area zone } AD \times \frac{1}{3}OA$ . § 186  
Q. E. D.



**987.** COR. I. Let  $H$  denote the altitude of the zone which forms the base of the spherical sector.

$$\begin{aligned} \text{Then} \quad \text{vol. sph. sector} &= 2\pi RH \times \frac{1}{3}R \\ &= \frac{2}{3}\pi R^2 H. \end{aligned}$$

**988.** COR. II. *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*

*Hint.*—A sphere may be regarded as a spherical sector whose base is the surface of the sphere.

**989.** COR. III. *If  $V$  is the volume of a sphere,  $R$  its radius, and  $D$  its diameter,*

$$V = 4\pi R^3 \times \frac{1}{3}R = \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3.$$

**990.** COR. IV. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

**991.** COR. V. *The volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.*

*Hint.*—Let  $v$  be the volume of the spherical pyramid,  $s$  the area of its base, and  $R$  the radius of the sphere.

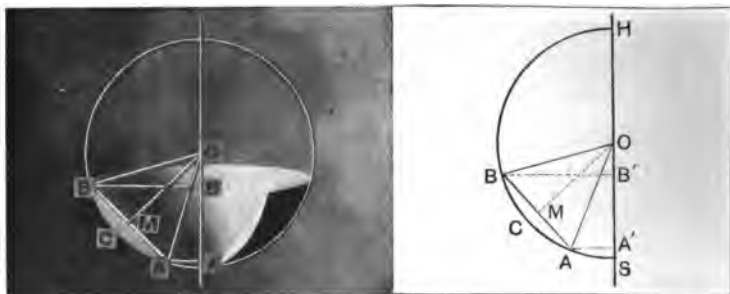
Also let  $V$  be the volume of the sphere and  $S$  the area of its surface.

$$\text{Then} \quad \frac{v}{V} = \frac{s}{S}. \quad \text{§ 899 XII}$$

$$\text{And} \quad V = S \times \frac{1}{3}R.$$

## PROPOSITION XIII. THEOREM

**992.** *The volume of the solid generated by a circular segment revolving about a diameter exterior to it is equal to one-sixth the area of the circle whose radius is the chord of the segment multiplied by the projection of that chord upon the axis.*



GIVEN—a circular segment  $ACB$  revolving about the diameter  $HS$ .

Let  $A'B'$  be the projection of  $AB$  upon  $HS$ .

TO PROVE  $\text{vol. } ACB = \frac{1}{6}\pi \overline{AB}^2 \times A'B'$ .

Draw the radii  $OA$ ,  $OB$ , and draw  $OM$  perpendicular to  $AB$ .

Then  $\text{vol. } ACB = \text{vol. sector } AOB - \text{vol. triangle } AOB$ .

$$\begin{aligned} \text{Now } \text{vol. sector } AOB &= \text{zone } ACB \times \frac{1}{6}OA && \S 986 \\ &= 2\pi OA \times A'B' \times \frac{1}{6}OA && \S 980 \\ &= \frac{2}{3}\pi \overline{OA}^2 \times A'B', \end{aligned}$$

$$\begin{aligned} \text{and } \text{vol. triangle } AOB &= \text{area } AMB \times \frac{1}{6}OM && \S 985 \\ &= 2\pi OM \times A'B' \times \frac{1}{6}OM && \S 977 \\ &= \frac{2}{3}\pi \overline{OM}^2 \times A'B'. \end{aligned}$$

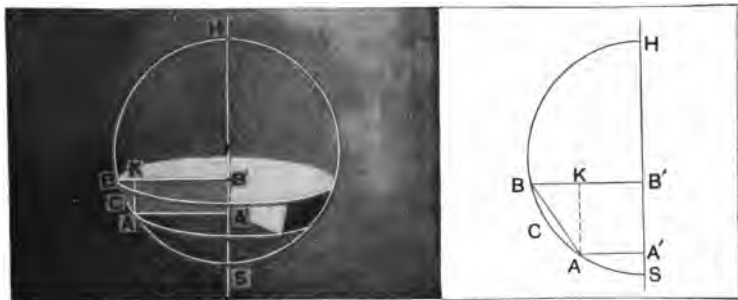
$$\text{Hence } \text{vol. } ACB = \frac{2}{3}\pi(\overline{OA}^2 - \overline{OM}^2) \times A'B'.$$

$$\text{But } \overline{OA}^2 - \overline{OM}^2 = \overline{AM}^2 = \frac{1}{4}\overline{AB}^2. \quad \S\S 318, 167$$

$$\text{Therefore } \text{vol. } ACB = \frac{1}{6}\pi \overline{AB}^2 \times A'B'. \quad \text{Q. E. D.}$$

PROPOSITION XIV. THEOREM

**993.** *The volume of a spherical segment is equal to half the sum of its bases multiplied by its altitude increased by the volume of a sphere whose diameter is equal to that altitude.*



GIVEN—a spherical segment, generated by the revolution of the figure  $ACBB'A'$  about the diameter  $HS$  of the semicircle  $HBS$ , the lines  $AA'$  and  $BB'$  generating the bases, and the arc  $ACB$  generating the curved surface of the segment. Denote  $BB'$  by  $r$ ,  $AA'$  by  $r'$ ,  $A'B'$  by  $h$ , and the volume of the spherical segment by  $V$ .

TO PROVE  $V = \frac{1}{2}(\pi r^2 + \pi r'^2)h + \frac{1}{8}\pi h^3$ .

The volume of the spherical segment is the sum of the volume generated by the circular segment  $ACB$  and the volume of the frustum of a cone generated by the trapezoid  $ABB'A'$ .

Hence  $V = \frac{1}{8}\pi \overline{AB}^2 \times h + \frac{1}{8}\pi(r^2 + r'^2 + rr')h$ . (I) §§ 992, 970

Draw  $AK$  perpendicular to  $BB'$ .

Then  $BK = r - r'$ .

Hence  $\overline{BK}^2 = r^2 + r'^2 - 2rr'$ .

Now  $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2 = h^2 + r^2 + r'^2 - 2rr'$ . § 317

Substituting this value for  $\overline{AB}^2$  in (I), we get

$$V = \frac{1}{2}(\pi r^2 + \pi r'^2)h + \frac{1}{8}\pi h^3, \quad \text{Q. E. D.}$$

**994. COR.** *The formula for the volume of a spherical segment of one base is*

$$V = \frac{1}{2}\pi r^2 h + \frac{1}{6}\pi h^3.$$



*Hint.*—This is obtained from the preceding formula by making the radius  $r'$  of one base equal to zero.

#### PROBLEMS OF DEMONSTRATION

**995. Exercise.**—The lateral area of a cylinder of revolution is equal to the area of a circle the radius of which is a mean proportional between the altitude of the cylinder and the diameter of its base.

**996. Exercise.**—The volume of a cylinder is equal to the product of the area of a right section by an element.

**997. Exercise.**—The area of a sphere is equal to the lateral area of a circumscribed cylinder of revolution.

**998. Exercise.**—The volume of a sphere is two-thirds the volume of a circumscribed cylinder of revolution.

**999. Exercise.**—If a cylinder of revolution of which the altitude is equal to the diameter of the base, and a cone of

revolution of which the slant height is equal to the diameter of the base, be inscribed in a sphere; the total area of the cylinder is a mean proportional between the area of the sphere and the total area of the cone, and the volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

**1000. Exercise.**—If a cylinder of revolution of which the altitude is equal to the diameter of the base, and a cone of revolution of which the slant height is equal to the diameter of the base, be circumscribed about a sphere; the total area of the cylinder is a mean proportional between the area of the sphere and the total area of the cone, and the volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

**1001. Exercise.**—Show that two cylinders of revolution, whose lateral areas are equal, are to each other as their radii, or inversely as their altitudes.

#### PROBLEMS FOR COMPUTATION

**1002.** (1.) A right section of a cylinder is a circle whose radius is 3 ft.; an element of the cylinder is 13 ft. Find the lateral area.

(2.) A cylindrical boiler is 12 ft. long and 6 ft. in diameter. Find its surface, and the number of gallons of water it will hold.

(3.) A cylindrical pail is 6 in. deep and 7 in. in diameter. Find its contents and the amount of tin required for its construction.

(4.) Find the volume generated by a rectangle 9 dcm. long and 4 dcm. broad (*a*) in revolving about its longer side; (*b*) in revolving about its shorter side.

(5.) A conical cistern is 13 in. deep and 12 in. across the top, which is circular. Find its contents.

(6.) A conical church steeple is 50 ft. high and 10 ft. in diameter at the base. How much would it cost to paint the steeple at 10 cents a square foot?

(7.) A cube, an edge of which is 2 in., is inscribed in a cone of revolution, of which the altitude is 5 in. Find the volume of the cone.

(8.) The sides of an equilateral triangle are each 10 in. What is the volume generated, if the triangle revolve about its altitude? What is the area of the surface generated?

(9.) The sides of a triangle are each 12.49 in. What is the volume generated if the triangle revolve about one side? What is the area of the surface generated?

(10.) Find the total area of a frustum of a cone of revolution, the radii of whose bases are 12 cm. and 7 cm., and whose altitude is 9.7 cm.

(11.) Find the volume of a frustum of a circular cone, the areas of whose bases are 12 sq. in. and 8 sq. in., and whose altitude is 5 in.

(12.) If the radius of a sphere is 647 cm.,

(a.) What is its area?

(b.) What is its volume?

(c.) What is the area of a lune whose angle is  $35^\circ$ ?

(d.) What is the volume of the spherical ungula whose base is the preceding lune?

(e.) What is the area of a spherical polygon whose angles are  $140^\circ$ ,  $65^\circ$ ,  $120^\circ$ ,  $50^\circ$ ?

(f.) What is the volume of the spherical pyramid whose base is the preceding spherical polygon.

(13.) A sphere and a cylinder of revolution have equal

areas. What is the ratio of the area of a sphere of half the diameter to the area of a similar cylinder of two-thirds the altitude?

(14.) How many marbles  $\frac{3}{4}$  in. in diameter can be made from 100 cu. in. of glass, if there is no waste in melting?

(15.) Assuming the earth to be a sphere 7960 miles in diameter, what is the area of its surface? What is its volume?

(16.) The surface of a sphere is 1.514 sq. m. Find its radius.

(17.) The volume of a sphere is 1000 cu. in. Find its radius.

(18.) The surface of a sphere is 4632 sq. m. Find its volume.

(19.) Show that, if  $S$  is the surface of a sphere and  $V$  its volume,

$$36\pi V^2 = S^3.$$

(20.) A hollow rubber ball is 2 in. in diameter and the rubber is  $\frac{3}{16}$  in. thick. How much rubber would be used in the manufacture of 1000 such balls?

(21.) If a sphere of iron weighs 999 lbs., how much would a sphere of iron of one third the diameter weigh? 990

(22.) A cone of revolution and a cylinder of revolution each have as base a great circle of a sphere, and as altitude the radius of the sphere. Find the ratios of the total surfaces of the cone and cylinder to the surface of the sphere.

(23.) Find the ratio of the volumes of a cone of revolution and a cylinder of revolution to the volume of a sphere, if the bases of the cone and cylinder are each equal to a great circle of the sphere, and the altitudes of the cone and cylinder are each equal to the diameter of the sphere.



(24.) Find the total area and the volume of the cylinder and of the cone in Problem (22), if the radius of the sphere is 1 dcm.

(25.) Find the total area and the volume of the cylinder and of the cone in Problem (23), if the radius of the sphere is 59.77 cm.

(26.) If the radius of a sphere is 4.581 in., what is the area of a zone whose altitude is 1.456 in.?

(27.) A dome is in the form of a spherical zone of one base, and its height is 30 ft. Find its surface if the radius of the sphere is 35 ft.

(28.) The radius of a sphere is 6.742 in.; the altitude of a zone is 2 in. Find the volume of the spherical sector of which this zone is the base.

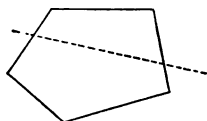
(29.) Find the volume of a spherical segment of one base whose altitude is 3 cm., and the radius of whose base is 9.643 cm.

# PLANE GEOMETRY

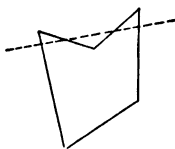
## APPENDIX

**1003. Def.**—A polygon is **convex**, if no straight line can meet its perimeter in more than two points. [Repeated from § 65.]

**1004. Def.**—A polygon is **re-entrant**, if a straight line can be drawn meeting its perimeter in more than two points.

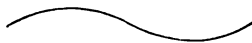


CONVEX POLYGON



RE-ENTRANT POLYGON

**1005. Def.**—A **curved line**, or **curve**, is a line, no part of which is straight. [Repeated from § 765.]

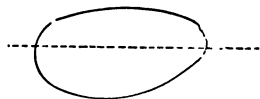


CURVED LINES

**1006. Def.**—A curve is **closed**, if it returns upon itself.

**1007. Def.**—A closed curve is **convex**, if no straight line can meet it in more than two points.

**1008. Def.**—A closed curve is **re-entrant**, if a straight line can be drawn meeting it in more than two points.



CONVEX CLOSED CURVE



RE-ENTRANT CLOSED CURVE

## PROPOSITION I. THEOREM

**1009.** *The circumference of a circle is a convex curve.*

For, if a straight line could meet it in three points, we would have three equal straight lines (the radii to the points of intersection) from a point to a straight line. It follows from § 100 that this is impossible.

**1010.** *Def.*—A **secant** of a convex closed curve is a straight line meeting it in two points.

**1011.** *Def.*—A **tangent** to a convex closed curve is a straight line meeting it in only one point, however far the line is produced.



SECANT



TANGENT

## PROPOSITION II. THEOREM

**1012.** *A tangent can be drawn at any point of a convex closed curve.*

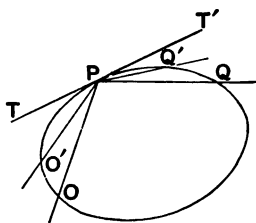


FIG. 1

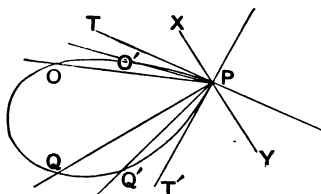


FIG. 2

If we imagine a secant  $PO$  of a convex closed curve to revolve about one point of intersection  $P$  as a pivot, while the intersection  $O$  moves along the curve toward  $P$ , the limiting position of this secant, when  $O$  coincides with  $P$ , will be a tangent  $PT$  to the curve at  $P$ .

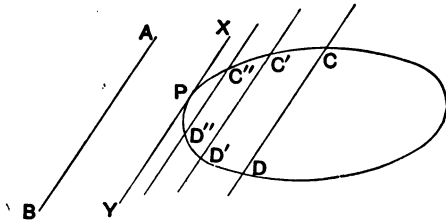
Q. E. D.

**1013. Remark.**—If we suppose a secant  $PQ$  to revolve in the opposite direction, so that  $Q$  approaches  $P$  on the other side, its limiting position will in general be another tangent  $PT'$  at the point  $P$  (Fig. 2). In this case there will be an infinite number of other tangents, as  $XY$ . These will all lie in the angle through which  $PT$  must turn, in the direction in which  $PO$  revolved, to coincide with  $PT'$ . In what follows we will suppose that the two tangents,  $PT$  and  $PT'$ , are the same straight line, as in Fig. 1, thus excluding from consideration curves like that in Fig. 2.

That is, to the curves here considered, only one tangent can be drawn at a given point.

PROPOSITION III. THEOREM

**1014.** Two tangents can be drawn to a convex closed curve parallel to a given straight line.



Draw a secant  $CD$  parallel to the given straight line  $AB$ .

Suppose  $CD$  to move in a direction perpendicular to  $AB$ , but always remaining parallel to  $AB$ .

The points  $C$  and  $D$  will move along the curve and will ultimately come together so as to coincide.

If this limiting position of  $CD$  is  $XY$ ,  $P$  being the point in which  $C$  and  $D$  coincide, then  $XY$  is tangent to the curve at  $P$  and is parallel to  $AB$ .

If  $CD$  had moved in the opposite direction we would have obtained another tangent to the curve parallel to  $AB$ . Hence we see that two tangents can be drawn to a convex closed curve parallel to a given straight line.

Q. E. D.

## PROPOSITION IV. THEOREM

**1015.** *A convex closed curve is greater than the perimeter of any inscribed polygon.*

*Hint.*—The proof is identical with that of Proposition V., Book V.

## PROPOSITION V. THEOREM

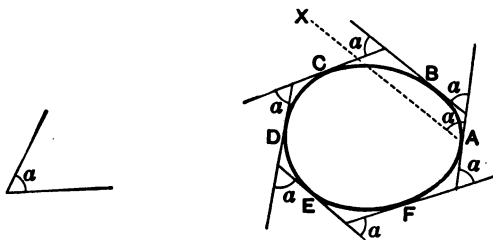
**1016.** *A convex closed curve is less than the perimeter of a circumscribed polygon or any enveloping line.*

*Hint.*—The proof is identical with that of Proposition VI., Book V.

**1017.** *Def.*—A polygon is equiangular when its angles are all equal.

## PROPOSITION VI. LEMMA

**1018.** *About any given convex closed curve an equiangular polygon of any required number of sides can be circumscribed.*



GIVEN

$ABCDEF$ , any convex closed curve.

TO PROVE—an equiangular polygon of  $n$  sides can be circumscribed.

Take the angle  $a$  one- $n^{\text{th}}$  of four right angles.

At any point  $A$  on the circumference draw a tangent, and from  $A$  draw the secant  $AX$ , making the angle  $a$  with the tangent.

Draw a tangent to the curve parallel to  $AX$ , and let the point of tangency be  $B$ .

Then the tangent at  $B$  makes the angle  $a$  with the tangent at  $A$ . § 49

In like manner draw a tangent at  $C$ , making the angle  $a$  with the second tangent, and so proceed until  $n$  tangents are drawn.

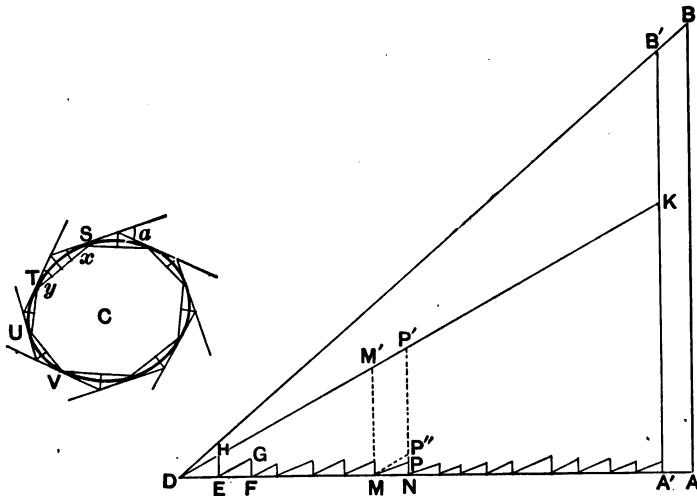
The last tangent must form in like manner the angle  $a$  with the first tangent, otherwise the sum of the  $n$  exterior angles of the polygon would not equal four right angles. § 69

Since the exterior angles are all equal, their supplementary angles, the angles of the polygon, must be all equal.

Hence the polygon is equiangular and of  $n$  sides. Q. E. D.

PROPOSITION VII. THEOREM

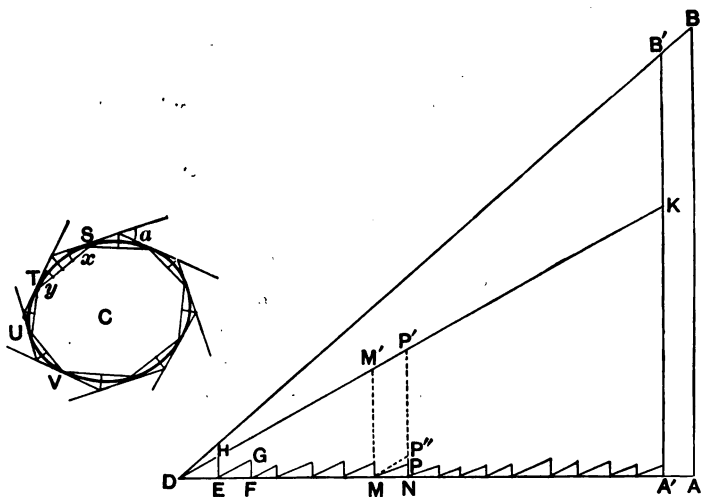
1019. *The circumference of a convex closed curve is the limit which the perimeters of a series of inscribed and circumscribed polygons approach when the number of their sides is indefinitely increased; and the area of the curve is the limit of the areas of these polygons.*



GIVEN any convex closed curve  $C$ .

TO PROVE—I. Its circumference is the common limit which the perimeters of a series of inscribed and circumscribed polygons approach when the number of their sides is indefinitely increased.

II. The area of the curve is the common limit which the areas of the inscribed and circumscribed polygons approach when the number of their sides is indefinitely increased.



Draw  $AB$  of any assigned length, no matter how small. We shall prove that the difference between the inscribed and circumscribed perimeters can be made less than  $AB$ .

Conceive the straight line  $AD$ , equal in length to the circumference of  $C$ , to be drawn perpendicular to  $AB$ . Join  $BD$ .

Now divide 4 right angles into such a number of equal parts that each part  $a$  shall be less than the angle  $BDA$ . Let  $n$  be the number of parts.

Circumscribe about  $C$  an equiangular polygon of  $n$  sides whose exterior angle is  $a$ . § 1018

Join the points of tangency, forming an inscribed polygon of  $n$  sides.

From the vertices of the circumscribed polygon draw perpendiculars to the sides of inscribed polygon, thus forming  $2n$  right triangles.

The angles at  $S, T, U, V$ , etc., between the tangents and chords, as  $x$  or  $y$ , are each less than  $a$  (§ 59), and therefore still less than angle  $ADB$ .

Of these angles  $x, y$ , etc., select the greatest, and place the right triangle containing it within the angle  $ADB$  in the position  $DEH$ .



In like manner place all the other right triangles along  $DA$ , as  $EFG$ , etc., irrespective of order. Let the last one extend to  $A'$ .

Thus the sum of the bases, or  $DA'$ , equals the inscribed perimeter (less than  $DA$ ), and the sum of the hypotenuses equals the circumscribed perimeter.

Produce  $DH$  to meet the perpendicular  $A'B'$  at  $K$ .

I. Any hypotenuse, as  $MP$ , can easily be proved less than  $M'P'$ , the portion of  $DK$  included between perpendiculars at  $M$  and  $N$ . § 99

Hence, adding all such inequalities,  $DK$  is greater than the sum of the hypotenuses  $DH, EG$ , etc.

That is,  $DK$  is greater than the circumscribed perimeter.

Now  $DA'$  is equal to the inscribed perimeter.

Hence the difference of the circumscribed and inscribed perimeters is less than  $DK - DA'$ .

But  $DK - DA'$  is less than  $KA'$ . § 137

And  $KA'$  is less than  $A'B'$ . And  $A'B'$  is less than  $AB$ .

Much more, therefore, is the difference of the perimeters less than  $AB$ .

We can thus make the difference of perimeters as small as we please.

But the circumference is always intermediate between the perimeters.

Hence either perimeter can be made to differ from the circumference by less than any assigned quantity.

Therefore the circumference is the common limit to which the perimeters approach. Q. E. D.

II. Moreover, the difference between the areas of the inscribed and circumscribed polygons consists of the  $2n$  right triangles, which is less than the triangle  $ADB$ .

But since the base  $AD$  of this triangle is constant we can make its area as small as we please by making its altitude  $AB$  as small as we please. § 187

Hence the difference between the polygons can be made as small as we please.

But the area of the curve is always intermediate between the polygons.



Hence either polygon can be made to differ from the area of the curve by less than any assigned quantity.

Therefore the area of the curve is the common limit to which the polygon-areas approach. Q. E. D.

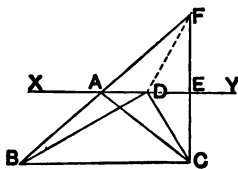
#### MAXIMA AND MINIMA OF PLANE FIGURES

**1020. Def.**—Of the values which a variable quantity assumes, the largest value is called the **maximum**; the smallest, the **minimum**.

Thus, the diameter of a circle is the maximum among all straight lines joining two points of the circumference; and among all the lines drawn from a given point to a given straight line the perpendicular is the minimum.

#### PROPOSITION VIII. THEOREM

**1021.** *Of all triangles having the same base and equal areas, that which is isosceles has the minimum perimeter.*



**GIVEN**—the isosceles triangle  $ABC$  and any other triangle  $DBC$  having an equal area and the same base  $BC$ .

**TO PROVE**—the perimeter of  $ABC$  is less than the perimeter of  $DBC$ .

*Outline proof.*—The vertices  $A$  and  $D$  are in the straight line  $XY$  parallel to  $BC$ . (Why?)

Draw  $CE$  perpendicular to  $BC$ , meeting  $BA$  produced at  $F$ . Join  $DF$ .

The angle  $CAE = \text{angle } FAE$ , and the triangle  $CAE = \text{triangle } FAE$ .

Hence  $AE$  is perpendicular to  $CF$  at its middle point.

Now  $AB + AF < DB + DF$ .

Or  $AB + AC < DB + DC$ .

Hence  $BC + AB + AC < BC + DB + DC$ .

Q. E. D.

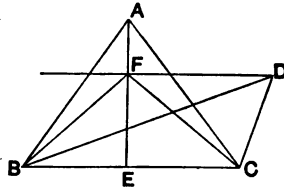
**1022. Remark.**—The converse of the preceding proposition is also true, viz.: *Of all triangles having the same base and equal areas, that which has the minimum perimeter is isosceles.* In fact, it is practically the same theorem as the proposition itself, for there is only one isosceles triangle fulfilling the given conditions, and only one triangle of minimum perimeter fulfilling the given conditions; just as to say that John Smith is the tallest man in the room is equivalent to saying that the tallest man in the room is John Smith, provided we know that there is only one John Smith in the room and only one tallest man.

**1023. COR.** *Of all triangles having the same area, that which is equilateral has the minimum perimeter.*

**1024. Def.**—When two figures have equal perimeters they are called **isoperimetric**.

PROPOSITION IX. THEOREM

**1025.** *Of all isoperimetric triangles having the same base, that which is isosceles has the maximum area.*



GIVEN—the isosceles triangle  $ABC$  and *any* other triangle  $DBC$  having an equal perimeter and the same base  $BC$ .

TO PROVE the area of  $ABC >$  area  $DBC$ .

*Outline proof.*—Draw  $AE$  perpendicular to  $BC$  and  $DF$  parallel to  $BC$ .

Join  $FB$  and  $FC$ .

The triangles  $FBC$  and  $DBC$  have equal areas.

But  $FBC$  is isosceles.

Therefore perimeter  $FBC <$  perimeter  $DBC$ .

Or perimeter  $FBC <$  perimeter  $ABC$ .

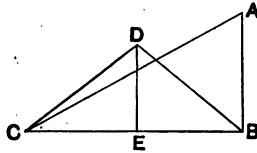
Hence  $BF < BA$ , and  $FE < AE$ .

Therefore the area of triangle  $ABC >$  area of triangle  $DBC$ . Q. E. D.

**1026.** COR. *Of all isoperimetric triangles, that which is equilateral has the maximum area.*

PROPOSITION X. THEOREM

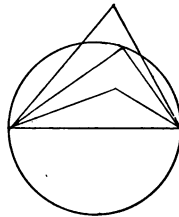
**1027.** *Of all triangles having two sides of one equal to two sides of the other, that in which these two sides are perpendicular to each other is the maximum.*



The proof is left to the student.

PROPOSITION XI. THEOREM

**1028.** *The locus of the vertex of a right angle whose sides pass through two fixed points is the circumference of a circle whose diameter is the straight line joining those points.*



*Hint.*—Apply §§ 202, 207, 210.

PROPOSITION XII. THEOREM

**1029.** *Of all isoperimetric plane figures, the maximum figure is a circle.*

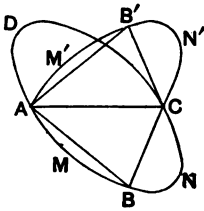


FIG. 1

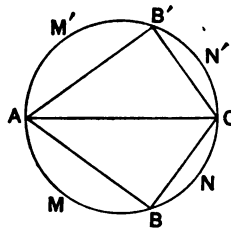


FIG. 2

GIVEN—among any number of isoperimetric plane figures the figure of maximum area  $ABCD$ .

TO PROVE that  $ABCD$  is a circle.

First, draw any straight line  $AC$  (Fig. 1), dividing its perimeter into two parts of equal length.

Then  $AC$  will also divide its surface into two parts of equal area.

For, if not, as for example if  $\text{area } ABC > \text{area } ADC$ , form the figure  $AB'C$  symmetrical to  $ABC$  by revolving  $ABC$  on  $AC$  as an axis.

Then the figure  $ABCB'A$  would be greater than  $ABCD$  and would have the same perimeter.

Hence  $ABCD$  would not be the maximum.

This would be contrary to the hypothesis.

Therefore  $AC$  does divide the surface into two equivalent parts.

Secondly, take any point  $B$  in the semiperimeter  $ABC$ .

We will prove that the angle  $ABC$  is a right angle.

Form the figure  $AB'C$  symmetrical to  $ABC$ .

Then  $\text{area } AB'C = \text{area } ABC$ .

But we have just proved  $\text{area } ABC = \text{area } ADC$ .

Therefore  $\text{area } ABCB'A = \text{area } ABCDA$ . Ax. 2

That is,  $ABCB'A$  is equivalent to the maximum figure.

Now, if the angle  $ABC$  were not a right angle, we could increase the area of the equal triangles  $ABC$  and  $AB'C$  by moving the points  $A$  and  $C$  nearer together or farther apart, so as to make  $ABC$  a right angle, without changing the lengths of the straight lines  $AB, BC, AB', B'C$ , and without changing the areas of the segments  $AMB, BNC, AM'B', B'N'C$ .

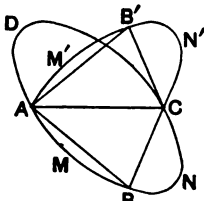


FIG. 1

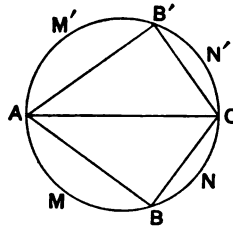


FIG. 2

In doing this the area of the whole figure  $ACB'A$  would be increased.

But this is impossible, since  $ACB'A$  is equivalent to the maximum figure.

Therefore  $ABC$  must be a right angle.

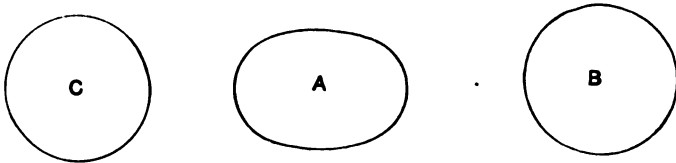
In like manner, if we should choose any point  $D$  in the semiperimeter  $ADC$ , we could show that  $ADC$  would be a right angle.

Therefore the figure  $ABCD$  is a circle.

§ 1028  
Q. E. D.

PROPOSITION XIII. THEOREM

**1030.** *Of all plane figures containing the same area, the circle has the minimum perimeter.*



GIVEN—a circle  $C$ , and any other figure  $A$  having the same area as  $C$ .

TO PROVE the perimeter of  $C$  is less than that of  $A$ .

Let  $B$  be a circle having the same perimeter as the figure  $A$ .

Then area  $A < \text{area } B$ , or area  $C < \text{area } B$ .

§ 1029

Now, of two circles, that which is the less has the less perimeter.

§§ 491, 499

Therefore the perimeter of  $C$  is less than the perimeter of  $B$ , or less than the perimeter of  $A$ .

Q. E. D.

PROPOSITION XIV. THEOREM

**1031.** *Of all the polygons constructed with the same given sides, that is the maximum which can be inscribed in a circle.*



GIVEN—a polygon  $P$  inscribed in a circle, and  $P'$ , any other polygon constructed with the same sides and not inscribable in a circle.

TO PROVE that  $P > P'$ .

Upon the sides of the polygon  $P'$  construct circular segments equal to those on the corresponding sides of  $P$ .

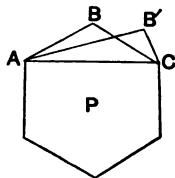
The whole figure  $S'$  thus formed has the same perimeter as the circle  $S$ .

Therefore area of  $S >$  area of  $S'$ . § 1029

Subtracting the circular segments from both,  
 $P > P'$ . Q. E. D.

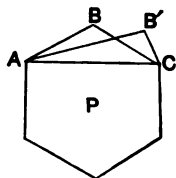
PROPOSITION XV. THEOREM

**1032.** *Of all isoperimetric polygons having the same number of sides the maximum is a regular polygon.*



GIVEN— $P$  the maximum of all the isoperimetric polygons of the same number of sides.

TO PROVE that  $P$  is a regular polygon.



If two of its sides, as  $AB'$ ,  $B'C$ , were unequal, the isosceles triangle  $ABC$ , having the same perimeter as  $AB'C$  and a greater area, could be substituted for the triangle  $ABC$ . § 1025

This would increase the area of the whole polygon without changing the length of the perimeter or the number of its sides.

Hence the sides of the maximum polygon must be all equal.

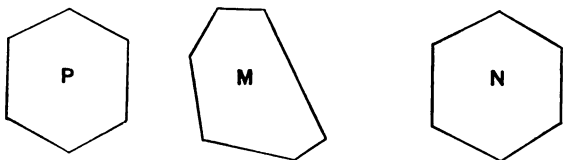
But the maximum of all polygons constructed with the same given sides must be inscriptible in a circle. § 1031

Therefore  $P$  is a regular polygon.

§§ 164, 469  
Q. E. D.

PROPOSITION XVI. THEOREM

**1033.** *Of all polygons having the same number of sides and the same area, the regular polygon has the minimum perimeter.*



GIVEN— $P$ , a regular polygon, and  $M$ , any other polygon of the same number of sides and area as  $P$ .

TO PROVE the perimeter of  $P <$  perimeter of  $M$ .

Let  $N$  be a regular polygon having same perimeter and same number of sides as  $M$ .

Then area  $M <$  area  $N$ , or area  $P <$  area  $N$ . § 1032

But of two regular polygons of the same number of sides the one of less area has the less perimeter. §§ 482, 483

Therefore the perimeter  $P$  is less than that of  $N$ , or less than that of  $M$ . Q. E. D.

# EXERCISES

## BOOK I

### PROBLEMS OF DEMONSTRATION

**1.** The bisector of an angle of a triangle is less than half the sum of the sides containing the angle.

**2.** The median drawn to any side of a triangle is less than half the sum of the other two sides, and greater than the excess of that half sum above half the third side.

**3.** The shortest of the medians of a triangle is the one drawn to the longest side.

**4.** The sum of the three medians of a triangle is less than the sum of the three sides, but greater than half their sum.

**5.** In any triangle the angle between the bisector of the angle opposite any side and the perpendicular from the opposite vertex on that side is equal to half the difference of the angles adjacent to that side.

**6.**  $LM$  and  $PR$  are two parallels which are cut obliquely by  $AB$  in the points  $A, B$ , and at right angles by  $AC$  in the points  $A, C$ ; the line  $BED$ , which cuts  $AC$  in  $E$  and  $LM$  in  $D$ , is such that  $ED$  is equal to  $2AB$ . Prove that the angle  $DBC$  is one-third the angle  $ABC$ .

**7.** The sum of the diagonals of a quadrilateral is less than the sum of the four lines joining any point other than the intersection of the diagonals to the four vertices.

**8.** The difference between the acute angles of a right triangle is equal to the angle between the median and the perpendicular drawn from the vertex of the right angle to the hypotenuse.

**9.** In a right triangle the bisector of the right angle also bisects the angle between the perpendicular and the median from the vertex of the right angle to the hypotenuse.



**10.** In the triangle formed by the bisectors of the exterior angles of a given triangle, each angle is one-half the supplement of the opposite angle in the given triangle.

**11.** A right triangle can be divided into two isosceles triangles.

**12.** A median of a triangle is greater than, equal to, or less than half of the side which it bisects, according as the angle opposite that side is acute, right, or obtuse.

**13.** The point of intersection of the perpendiculars erected at the middle of each side of a triangle, the point of intersection of the three medians, and the point of intersection of the three perpendiculars from the vertices to the opposite sides are in a straight line; and the distance of the first point from the second is half the distance of the second from the third.

**14.** Find the locus of a point the sum or the difference of whose distances from two fixed straight lines is given.

**15.** On the side  $AB$ , produced if necessary, of a triangle  $ABC$ ,  $AC'$  is taken equal to  $AC$ ; similarly on  $AC$ ,  $AB'$  is taken equal to  $AB$ , and the line  $B'C'$  drawn to cut  $BC$  in  $P$ . Prove that the line  $AP$  bisects the angle  $BAC$ .

**16.** The point of intersection of the straight lines which join the middle points of opposite sides of a quadrilateral is the middle point of the straight line joining the middle points of the diagonals.

**17.** The angle between the bisector of an angle of a triangle and the bisector of an exterior angle at another vertex is equal to half the third angle of the triangle.

**18.** If  $L$  and  $M$  are the middle points of the sides  $AB$ ,  $CD$  of a parallelogram  $ABCD$ , the straight lines,  $DL$ ,  $BM$  trisect the diagonal  $AC$ .

**19.**  $ABC$  is an equilateral triangle;  $BD$  and  $CD$  are the bisectors of the angles at  $B$  and  $C$ . Prove that lines through  $D$  parallel to the sides  $AB$  and  $AC$  trisect  $BC$ .

**20.** The angle between the bisectors (produced only to their point of intersection) of two adjacent angles of a quadrilateral is equal to half the sum of the two other angles of the quadrilateral. The acute angle between the bisectors of two opposite angles of a quadrilateral is equal to half the difference of the other angles.

**21.** The bisectors of the angles of a quadrilateral form a second quadrilateral of which the opposite angles are supplementary. When the first quadrilateral is a parallelogram, the second is a rectangle whose diagonals are parallel to the sides of the parallelogram and each equal to the difference of two adjacent sides of the parallelogram. When the first quadrilateral is a rectangle, the second is a square.

**22.** Two quadrilaterals are equal if an angle of the one is equal to an angle of the other, and the four sides of the one are respectively equal to the four similarly situated sides of the other.

**23.** If two polygons have the same number of sides and this number is odd, and if one polygon can be placed upon the other so that the middle points of the sides of the first fall upon the middle points of the sides of the second, the polygons are equal.

#### PROBLEMS OF CONSTRUCTION

**24.** Find a point in a straight line such that the sum of its distances from two fixed points on the same side of the straight line shall be the least possible.

**25.** Find a point in a straight line such that the difference of its distances from two fixed points on opposite sides of the line shall be the greatest possible.

**26.** Draw through a given point within a given angle a straight line such that the part intercepted between the sides of the angle shall be bisected by the given point.

**27.** Through a given point without a straight line to draw a straight line making a given angle with the given line.

**28.** Divide a rectangle 7 in. long and 3 in. broad into three figures which can be joined together so as to form a square.

## BOOK II

#### PROBLEMS OF DEMONSTRATION

**29.** If a circle is circumscribed about an equilateral triangle and from any point in the circumference straight lines are drawn to the three vertices, one of these lines is equal to the sum of the other two.

**30.** If one circle touches another internally at  $P$  and a tangent to the first at  $Q$  intersects the second in  $M, N$ , then the angles  $MPQ, NPQ$  are equal.

**31.** The centre of one circle is on the circumference of another; if  $A$  and  $B$  are the points in which the common tangents touch the second, prove that the line  $AB$  is tangent to the first.

**32.** The trapezoid of which the non-parallel sides are equal is the only trapezoid which can be inscribed in a circle.

**33.** From any point on the circumference of a circle circumscribed about an equilateral triangle  $ABC$ , straight lines are drawn parallel respectively to  $BC, CA, AB$ , meeting the sides  $CA, AB, BC$  at  $M, N, O$ . Prove that  $M, N, O$  are in the same straight line.

**34.** If a quadrilateral be inscribed in a circle and the opposite sides produced to meet at  $M$  and  $N$ , prove that the bisectors of the angles at  $M$  and  $N$  meet at right angles.

**35.** Two circles pass through the vertex and a point in the bisector of an angle. Prove that the portions of the sides of the angle intercepted between their circumferences are equal.

**36.** Each angle formed by joining the feet of the perpendiculars of a triangle is bisected by the perpendicular from the opposite vertex.

**37.** Circumscribe a circle about a triangle; from one vertex drop a perpendicular on the opposite side to meet it in  $M$ , and produce to meet the circumference in  $N$ . Then, if  $P$  is the intersection of the perpendiculars,  $PM = MN$ .

**38.** A fixed circle touches a fixed straight line; any circle is drawn touching the fixed circle at  $B$  and the fixed straight line at  $C$ . Prove that the straight line  $BC$  passes through a fixed point.

**39.** The distance from the centre of the circle circumscribed about a triangle to a side is equal to half the distance from the opposite vertex to the intersection of the three perpendiculars from the vertices to the sides.

**40.** Prove that the straight lines joining the vertices of a triangle with the opposite points of tangency of the inscribed circle meet in a point.

**41.** If two points are given on the circumference of a given circle, another fixed circle can be found such that if any two lines be drawn from the given points to intersect on its circumference, the straight line joining the points in which these lines meet the given circle a second time will be of constant length.

**42.** If the three diagonals joining the opposite vertices of a hexagon are equal and the opposite sides are parallel in pairs, the hexagon can be inscribed in a circle.

**43.** Equilateral triangles are constructed on the sides of a given triangle and external to it. Prove that the three lines, each joining the outer vertex of one of the equilateral triangles to the opposite vertex of the given triangle, meet in a point and are equal.

**44.** On each side of a triangle construct an isosceles triangle with the adjacent angles equal to  $30^\circ$ . Prove that the straight lines joining the outer vertices of these three triangles are equal.

## LOCI

**45.** One side and the opposite angle of a triangle are given, and equilateral triangles are constructed on the other two (variable) sides. Find the locus of the middle point of the straight line joining the outer vertices of the equilateral triangles.

**46.** Through a vertex of an equilateral triangle is drawn any straight line  $PQ$ , terminated by the perpendiculars to the opposite side erected at the extremities of that side; on  $PQ$  as a side a second equilateral triangle is constructed. Find the locus of the opposite vertex of the second equilateral triangle.

**47.** The sides of a right triangle are given in position, its hypotenuse in length. Find the locus of the middle point of the hypotenuse.

**48.**  $AC$ ,  $BD$ , are fixed diameters of a circle, at right angles to each other, and  $P$  is any point on the circumference.  $PA$  cuts  $BD$  in  $E$ ;  $EF$ , parallel to  $AC$ , cuts  $PB$  in  $F$ . Prove that the locus of  $F$  is a straight line.

## PROBLEMS OF CONSTRUCTION

**49.** Draw four circles through a given point and tangent to two given circles.

**50.** Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the point and the given line may be equal to the part within the given circle.

**51.** Find a point in a given straight line such that tangents from it to two given circles shall be equal.

**52.** Construct a right triangle, having given one side and the perpendicular from the vertex of the right angle on the hypotenuse.

**53.** The distances from a point to the three nearest corners of a square are 1 in., 2 in.,  $2\frac{1}{2}$  in. Construct the square.

**54.** Construct a right triangle, having given the medians from the extremities of the hypotenuse.

**55.** Construct a right triangle, having given the difference between the hypotenuse and each side.

**56.** Construct a triangle, having given one angle and the medians drawn from the vertices of the other angles.

**57.** Construct a triangle, having given an angle, the perpendicular from its vertex on the opposite side, and the sum of the sides including that angle.

**58.** Having given two concentric circles, draw a chord of the larger circle, which shall be divided into three equal parts by the circumference of the smaller circle.

**59.** Inscribe in a circle a quadrilateral  $ABCD$ , having the diagonal  $AC$  given in direction, the diagonal  $BD$  in magnitude, and having given the position of the point  $E$  in which the sides  $AB$  and  $CD$  meet when produced.

**60.** Draw a chord of given length through a given point, within or without a given circle.

**61.** Construct an equilateral triangle such that one vertex is at a given point, and the other two vertices are on a given straight line and a given circumference respectively.

## BOOK III

## PROBLEMS OF DEMONSTRATION

**62.** If from a given point straight lines are drawn to the extremities of any diameter of a given circle, the sum of the squares of these lines will be constant.

**63.** The straight line joining the middle of the base of a triangle to the middle point of the line drawn from the opposite vertex to the point at which the inscribed circle touches the base, passes through the centre of the inscribed circle.

**64.** The square of the straight line joining the centre of a circle to to any point of a chord plus the product of the segments of the chord is equal to the square of the radius.

**65.**  $P$  and  $Q$  are two points on the circumscribing circle of the triangle  $ABC$ , such that the distance of either point from  $A$  is a mean proportional between its distances from  $B$  and  $C$ . Prove that the difference between the angles  $PAB$ ,  $QAC$  is half the difference between the angles  $ABC$ ,  $ACB$ .

**66.** If a quadrilateral be circumscribed about a circle, prove that the middle points of its diagonals and the centre of the circle are in a straight line.

**67.** From the vertex of the right angle  $C$  of a right triangle  $ACB$  straight lines  $CD$  and  $CE$  are drawn, making the angles  $ACD$ ,  $ACE$  each equal to the angle  $B$ , and meeting the hypotenuse in  $D$  and  $E$ . Prove that  $\overline{DC}^2 : \overline{DB}^2 = AE : EB$ .

**68.**  $ABCD$  is a parallelogram; the circle through  $A$ ,  $B$ , and  $C$  cuts  $AD$  in  $A'$ , and  $DC$  in  $C'$ . Prove that

$$A'D : A'C' = A'C : A'B.$$

**69.** If two intersecting chords are drawn in a semicircle from the extremities of the diameter, the sum of the products of the segment adjacent to the diameter in each by the whole chord is equal to the square of the diameter.

**70.** If a quadrilateral circumscribe a circle the two diagonals and the two lines joining the points where the opposite sides of the quadrilateral touch the circle will all four meet in a point.

**71.** There are two points whose distances from three fixed points are in the ratios  $p : q : r$ . Prove that the straight line joining them passes through a fixed point whatever be the values of the ratios.

**72.** The lines joining the vertices of an equilateral triangle  $ABC$  to any point  $D$  meet the circumscribing circle in the points  $A', B', C$ . Prove that  $AD \cdot AA' + BD \cdot BB' + CD \cdot CC' = 2\overline{AB}^2$ .

**73.** If from any point perpendiculars are drawn to all the sides of a polygon, the two sums of the squares of the alternate segments of the sides are equal.

**74.** One circle touches another internally, and a third circle whose radius is a mean proportional between their radii passes through the point of contact. Prove that the other intersections of the third circle with the first two are in a line parallel to the common tangent of the first two.

**75.** If two tangents are drawn to a circle at the extremities of a diameter, the portion of any third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

**76.** A straight line  $AB$  is divided harmonically at  $P$  and  $Q$ ;  $M, N$  are the middle points of  $AB$  and  $PQ$ . If  $X$  be any point on the line, prove that  $XA \cdot XB + XP \cdot XQ = 2XM \cdot XN$ .

**77.** The radius of a circle drawn through the centres of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

**78.** The centres of the four escribed circles of a quadrilateral lie on the circumference of a circle.

**79.**  $O, O_1, O_2, O_3$  are the centres of the inscribed and three escribed circles of a triangle  $ABC$ . Prove that

$$AO \cdot AO_1 \cdot AO_2 \cdot AO_3 = \overline{AB}^2 \cdot \overline{AC}^2.$$

**80.** The opposite sides of a quadrilateral inscribed in a circle, when produced, meet at  $P$  and  $Q$ ; prove that the square of  $PQ$  is equal to the sum of the squares of the tangents from  $P$  and  $Q$  to the circle.

#### LOCI

**81.**  $A$  is a point on the circumference of a given circle,  $P$  a point

without the circle.  $AP$  cuts the circle again in  $B$ , and the ratio  $AP : AB$  is constant.

Find the locus of  $P$ .

**82.** Find the locus of a point whose distances from two given points are in a given ratio.

**83.** Find the locus of a point the sum of the squares of whose distances from the vertices of a given triangle is constant.

#### PROBLEMS OF CONSTRUCTION

**84.** Draw a circle such that, if straight lines be drawn from any point in its circumference to two given points, these lines shall have a given ratio.

**85.** Construct a triangle, having given the base, the line bisecting the opposite angle, and the diameter of the circumscribed circle.

**86.** Construct a right triangle, having given the difference between the sides and the difference between the hypotenuse and one side.

**87.** Construct a triangle, having given the perimeter, the altitude, and that one base angle is twice the other.

**88.** Construct a triangle, having given an angle, the length of its bisector, and the sum of the including sides.

**89.** From one extremity of a diameter of a given circle draw a straight line such that the part intercepted between the circumference and the tangent at the other extremity shall be of given length.

**90.** Divide a semi-circumference into two parts such that the radius shall be a mean proportional between the chords of the parts.

**91.** Construct a triangle, similar to a given triangle, such that two of its vertices may be on lines given in position, and its third vertex be at a given point.

**92.** Through four given points draw lines which will form a quadrilateral similar to a given quadrilateral.

**93.** Find a point such that its distances from three given points may have given ratios.

**94.** Divide a straight line harmonically in a given ratio.

**95.** A line perpendicular to the bisector of an angle of a triangle is drawn through the point in which the bisector meets the opposite



side. Prove that the segment on either of the other sides between this line and the vertex is a harmonic mean between those sides.

**96.** Draw through a given point within a circle a chord which shall be divided at that point in mean and extreme ratio.

#### PROBLEMS FOR COMPUTATION

**97.** (1.) The sides of a right triangle are 15 ft. and 18 ft. The hypotenuse of a similar triangle is 20 ft. Find its sides.

(2.) The sides of a right triangle are 16.213 in. and 32.426 in. Find the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse.

(3.) In an isosceles triangle the vertex angle is  $45^\circ$ ; each of the equal sides is 16 yds. Find the base in meters.

(4.) In a triangle whose sides are 247.93 mm., 641.98 mm., 521.23 mm., find the altitude upon the shortest side.

(5.) In a triangle whose sides are 4, 7, and 9, find the median drawn to the shortest side.

(6.) In a triangle whose sides are 123.41 in., 246.93 in., 157.62 in., compute the bisector of the largest angle.

(7.) Two adjacent sides of a parallelogram are 49 cm. and 53 cm. One diagonal is 58 cm. Find the other diagonal.

(8.) If the chord of an arc is 720 ft., and the chord of its half is 376 ft., what is the diameter of the circle?

(9.) From a point without a circle two tangents are drawn making an angle of  $60^\circ$ . The length of each tangent is 15 in. Find the diameter of the circle.

(10.) Find the radius of a circle circumscribing a triangle whose sides are 35.421 cm., 36.217 cm., 423.92 cm.

### BOOK IV

#### PROBLEMS OF DEMONSTRATION

**98.** A straight line  $AB$  is bisected in  $C$  and divided unequally in  $D$ . Prove that the sum of the squares on  $AD$  and  $DB$  is equal to twice the sum of the squares on  $AC$  and  $CD$ .

**99.** The area of a triangle is equal to the product of its three sides divided by four times the radius of its circumscribed circle.

**100.** Prove, by a geometrical construction, that the square on the hypotenuse of a right triangle is equal to four times the triangle plus the square on the difference of the sides.

**101.** Prove, by a geometrical construction, that the square on the hypotenuse of a right triangle is equal to the square on the sum of the sides minus four times the triangle.

**102.** On the side  $BC$  of the rectangle  $ABCD$  as diameter describe a circle. From its centre  $E$  draw the radius  $EG$  parallel to  $CD$  and in the direction  $C$  to  $D$ . Join  $G$  and  $C$  by a straight line cutting the diagonal  $BD$  in  $H$ . From  $H$  draw the line  $HK$  parallel to  $CD$  and in the direction  $C$  to  $D$ , cutting the circumference of the circle in  $K$ . Join  $BK$  and produce to meet  $CD$  in  $L$ . Then  $CL$  is the side of a square which is equivalent to the rectangle  $ABCD$ .

**103.** Construct any parallelograms  $ACDE$  and  $BCFG$  on the sides  $AC$  and  $BC$  of a triangle and exterior to the triangle. Produce  $ED$  and  $GF$  to meet in  $H$  and join  $HC$ ; through  $A$  and  $B$  draw  $AL$  and  $BM$  equal and parallel to  $HC$ . Prove that the parallelogram  $ALMB$  is equal to the sum of the parallelograms which have been constructed on the sides.

**104.** If similar triangles be circumscribed about and inscribed in a given triangle, the area of the given triangle is a mean proportional between the areas of the inscribed and circumscribed triangles.

**105.** Any fourth point  $P$  is taken on the circumference of a circle through  $A$ ,  $B$ , and  $C$ . Prove that the middle points of  $PA$ ,  $PB$ ,  $PC$  form a triangle similar to the triangle  $ABC$ , of one-fourth the area, and such that its circumscribing circle always touches the given circle at  $P$ .

**106.** Equilateral triangles are constructed on the four sides of a square all lying within the square. Prove that the area of the star-shaped figure formed by joining the vertex of each triangle to the two nearest corners of the square is equal to eight times the area of one of the equilateral triangles minus three times the area of the square.

**107.** A hexagon has its three pairs of opposite sides parallel. Prove that the two triangles which can be formed by joining alternate vertices are of equal area.

**108.** A quadrilateral and a triangle are such that two of the sides of the triangle are equal to the two diagonals of the quadrilateral and the angle between these sides is equal to the angle between the diagonals. Prove the areas of the quadrilateral and triangle are equal.

**109.** Prove that the straight lines drawn from the corners of a square to the middle points of the opposite sides taken in order form a square of one-fifth the area of the original square.

**110.** The area of the octagon formed by the straight lines joining each vertex of a parallelogram to the middle points of the two opposite sides is one-sixth the area of the parallelogram.

**111.**  $ABCD$  is a parallelogram. A point  $E$  is taken on  $CD$  such that  $CE$  is an  $n^{\text{th}}$  part of  $CD$ ; the diagonal  $AC$  cuts  $BE$  in  $F$ . Prove the following continued proportion connecting the areas of the parts of the parallelogram

$$ADEFA : AFB : BFC : CFE = n^2 + n - 1 : n^2 : n : 1$$

**112.** The squares  $ACKE$  and  $BCID$  are constructed on the sides of a right triangle  $ABC$ ; the lines  $AD$  and  $BE$  intersect at  $G$ ;  $AD$  cuts  $CB$  in  $H$ , and  $BE$  cuts  $AC$  in  $F$ . Prove that the quadrilateral  $FCHG$  and the triangle  $ABG$  are equivalent.

#### PROBLEMS OF CONSTRUCTION

**113.** Construct an equilateral triangle which shall be equal in area to a given parallelogram.

**114.** Construct a square which shall have a given ratio to a given square.

**115.** A pavement is made of black and white tiles, the black being squares, the white equilateral triangles whose sides are equal to the sides of the squares. Construct the pattern so that the areas of black and white may be in the ratio  $\sqrt{3} : 4$ .

**116.** Produce a given straight line so that the square on the whole line shall have a given ratio to the rectangle contained by the given line and its extension. When is the problem impossible?

**117.** Find a point in the base produced of a triangle such that a straight line drawn through it cutting a given area from the triangle may be divided by the sides of the triangle into segments having a given ratio.

**118.** Bisect a given quadrilateral by a straight line drawn through a vertex.

## PROBLEMS FOR COMPUTATION

**119.** (1.) If the area of an equilateral triangle is 164.51 sq. in., find its perimeter.

(2.) The perimeter of an equilateral pentagon is 25.135 ft. Its area is 23.624 sq. ft. Find the area of a similar pentagon one of whose sides is 10.361 ft.

(3.) Find, in acres, the area of a triangle, if two of its sides are 16.342 rds. and 23.461 rds., and the included angle is  $135^\circ$ .

(4.) Find the area of the triangle in the preceding example in hectares.

(5.) The sides of a triangle are 13.461, 16.243, and 20.042 miles. Find the areas of the parts into which it is divided by any median.

(6.) The sides of a triangle are 12 in., 15 in., and 17 in. Find the areas of the parts into which it is divided by the bisector of the smallest angle.

(7.) Two sides of a triangle are in the ratio 2 to 5. Find the ratio of the parts into which the bisector of the included angle divides the triangle.

(8.) The altitude upon the hypotenuse of a right triangle is 98.423 in. One part into which the altitude divides the hypotenuse is four times the other. Find the area of the triangle.

(9.) Find the perimeter of the triangle in the preceding example.

(10.) The areas of two similar polygons are 22.462 sq. in. and 14.391 sq. m. A side of the first is 2 in. Find the homologous side of the second.

(11.) The sides of a triangle are .016256, .013961, and .020202. Find the radius of the inscribed circle.

(12.) A mirror measuring 33 in. by 22 in. is to have a frame of uni-

form width whose area is to equal the area of the mirror; find what the width of the frame should be.

(13.) The sum of the radii of the inscribed, circumscribed, and an escribed circle of an equilateral triangle is unity. What is the area of the triangle?

## BOOK V

### PROBLEMS OF DEMONSTRATION

**120.** An equilateral polygon inscribed in a circle is regular. An equilateral polygon circumscribed about a circle is regular, if the number of sides is odd.

**121.** An equiangular polygon inscribed in a circle is regular if the number of sides is odd. An equiangular polygon circumscribed about a circle is regular.

**122.** The diagonals of a regular pentagon are equal.

**123.** The pentagon formed by the diagonals of a regular pentagon is regular.

**124.** An inscribed regular octagon is equivalent to a rectangle whose sides are equal to the sides of an inscribed and a circumscribed square.

**125.** If a triangle is formed having as sides the radius of a circle, the side of an inscribed regular pentagon, and the side of an inscribed regular decagon, this triangle will be a right triangle.

**126.** The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

**127.** If perpendiculars are drawn from the vertices of a regular polygon to any straight line through its centre, the sum of those which fall upon one side of the line is equal to the sum of those which fall upon the other side.

**128.** The area of any regular polygon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed polygons of half the number of sides.

**129.** If, on the sides of a right triangle as diameters, semi-circum-

ferences are described exterior to the triangle, and a circumference is drawn through the three vertices, the sum of the crescents thus formed is equivalent to the triangle.

**130.** If two circles are internally tangent to a third circle and the sum of their radii is equal to the radius of the third circle, the shorter arc of the third circle comprised between their points of contact is equal to the sum of the arcs of the two small circles from their points of contact with the third circle to their intersection which is nearest the large circle.

**131.** If  $CD$  is the perpendicular from the vertex of the right angle of a right triangle  $ABC$ , prove that the areas of the circles inscribed in the triangles  $ACD$ ,  $BCD$  are proportional to the areas of the triangles.

#### PROBLEMS OF CONSTRUCTION

**132.** To construct a circumference whose length shall equal the sum of the lengths of two given circumferences.

**133.** To construct a circle equivalent to the sum of two given circles.

**134.** To inscribe a regular octagon in a given square.

**135.** To inscribe a regular hexagon in a given equilateral triangle.

**136.** Divide a given circle into any number of parts proportional to given straight lines by circumferences concentric with it.

**137.** Find four circles whose radii are proportional to given lines, and the sum of whose areas is equal to the area of a given circle.

**138.** In a given equilateral triangle inscribe three equal circles each tangent to the two others and to two sides of the triangle.

**139.** In a given circle inscribe three equal circles each tangent to the two others and to the given circle.

**140.** The length of the circumference of a circle being represented by a given straight line, find approximately by a geometrical construction the radius.

#### PROBLEMS FOR COMPUTATION

**141.** (1.) A regular octagon is inscribed in a circle whose radius is 4 ft. Find the segment of the circle contained between one side of the octagon and its subtended arc.

(2.) Find the area of an equilateral triangle circumscribed about a circle whose radius is 14.361 in.

(3.) An isosceles right triangle is circumscribed about a circle whose radius is 3 cm. Find (a) each side; (b) its area; (c) the area in each corner of the triangle bounded by the circumference of the circle and two sides of the triangle.

(4.) Find the area of the circle inscribed in an equilateral triangle, one side of which is 7.4631 ft.

(5.) Find the difference between the area of a triangle whose sides are 4.6213 mm., 3.7962 mm., and 2.6435 mm., and the area of the circumscribed circle.

(6.) The area of a circle is 14632 sq. ft. Find its circumference in yards.

(7.) Find the area of a ring whose outer circumference is 15.437 ft., and whose inner circumference is 9.3421 ft.

(8.) Find the ratio of the areas of two circles inscribed in equilateral triangles, if the perimeter of one triangle is four times that of the other.

(9.) If the area of an equilateral triangle inscribed in a circle is 12 sq. ft., what is the area of a regular hexagon circumscribed about the same circle?

(10.) Find the side of a regular octagon whose area shall equal the sum of the areas of two regular hexagons, one inscribed in and the other circumscribed about a circle whose radius is 10.462 in.

(11.) A man has a circular farm 640 acres in extent. He gives to each of his four sons one of the four largest equal circular farms which can be cut off from the original farm. How much did each son receive?

(12.) A man has a circular tract of land 700 acres in area; he wills one of the three largest equal circular tracts to each of his three sons, the tract at the centre included between the three circular tracts to his daughter, and the tracts included between the circumference of the original tract and the three circular tracts to his wife. How much will each receive?

(13.) A man owned a tract of land 323,250 sq. m. in area, and in the

form of an equilateral triangle. To each of his three sons he gave one of the three largest equal circular tracts which could be formed from the given tract; to each of his three daughters one of the corner sections cut off by a circular tract; to each of his three grandchildren one of the side sections cut off by two of the circular tracts; he himself retained the central section included between the three circular tracts. Find the share of each.

## BOOK VI

## PROBLEMS OF DEMONSTRATION

**142.** If a straight line is parallel to a plane, the shortest distance of the line from all straight lines of the plane which are not parallel to it is the same.

**143.** If a straight line is parallel to a plane it is everywhere equidistant from the plane.

**144.** If a plane is passed through two vertices of a parallelogram, the perpendiculars to it from the other vertices are equal.

**145.** If from the foot of a perpendicular to a plane a straight line is drawn perpendicular to any line of the plane, and the intersection of these lines is joined to any point of the perpendicular to the plane, the last line will be perpendicular to the line of the plane.

**146.** The plane angle of a right dihedral angle is a right angle, and conversely. Two dihedral angles are to each other as their plane angles.

**147.** If a line is drawn in each face plane of any trihedral angle through its vertex and perpendicular to the opposite edge, prove that these three lines lie in the same plane.

**148.**  $A, B, C$  are points on the three edges of a trihedral angle of which the face angles are right angles;  $S$  is the projection of the vertex  $O$  on the plane  $ABC$ . Prove that the triangle  $AOB$  is a mean proportional between the triangles  $ABC$  and  $ASB$ .

## LOCI

**149.** Find the locus of a point in space the difference of the squares of whose distances from two given points is constant.



**150. Defs.**—The angle between two straight lines not in the same plane, that is, neither parallel nor intersecting, is the angle between two lines drawn through any point in space parallel respectively to the two lines and in the same directions.

Two straight lines in space are perpendicular when their angle, as defined above, is a right angle.

**151.** Find the locus of the middle point of a straight line of given length which has its extremities upon two given perpendicular but non-intersecting straight lines.

**152.** A straight line moves parallel to a fixed plane and intersects two fixed straight lines not in the same plane. Find the locus of a point which divides the part intercepted in a constant ratio.

**153.** Find the locus of a point in a given plane such that the straight lines joining it to two given points not in the plane make equal angles with the plane.

**154.** Find the locus of a point the sum of whose distances from two given planes is equal to a given straight line.

**155.** Find the locus of a point equidistant from the three faces of a triedral angle.

#### PROBLEMS OF CONSTRUCTION

**156.** Draw a line in a given plane, and through a given point in the plane, which shall be perpendicular to a given straight line in space.

**157.** Pass a plane cutting the faces of a polyedral angle of four faces in such a manner that the section shall be a parallelogram.

**158.** Given a straight line  $AB$  parallel to a plane  $M$ . From any point  $A$  in  $AB$  draw a straight line  $AX$ , of given length, to the plane  $M$ , so as to make the angle  $BAX$  equal to a given angle.

**159.** Through a given point in a plane, to draw a straight line in that plane which shall be at a given distance from a given point outside of the plane.

**160.** A given straight line intersects a given plane. Through the intersection draw a straight line in the given plane, making a given angle with the given line.

**161.** Given three straight lines in space. Draw a straight line from the first to the second parallel to the third.

**162.** Given two straight lines not in the same plane. Find a point in one at a given perpendicular distance from the other.

**163.** Through a given point draw a straight line to meet a given straight line and the circumference of a given circle not in the same plane with the given line.

## BOOK VII

### PROBLEMS OF DEMONSTRATION

**164.** A triangular pyramid is cut by a plane parallel to the base, and a plane is passed through each vertex of the base and the points where the cutting plane meets the two opposite lateral edges. Determine the locus of the point of intersection of the three planes thus passed.

**165.** At any point in the base of a regular pyramid a perpendicular to the base is erected, intersecting the lateral faces of the pyramid, or these faces produced. Prove that the sum of the perpendicular distances from the points of intersection to the base is constant.

**166.** The perpendicular from the centre of gravity of a tetraedron (§ 749) to any plane is one-fourth the sum of the four perpendiculars from the vertices of the tetraedron to the same plane.

**167.** If the edges of a hexaedron meet four by four in three points, the four diagonals of the hexaedron meet in a point.

**168.** Prove that straight lines through the middle points of the sides of any face of a tetraedron each parallel to the straight line connecting a fixed point  $D$  with the middle point of the opposite edge, meet in a point  $E$  such that  $DE$  passes through and is bisected by the centre of gravity of the tetraedron.

**169.** The sum of the perpendiculars drawn to the faces of a regular tetraedron from any point within is equal to the altitude of the tetraedron.

**170.** A regular octaedron is cut by a plane parallel to one of its faces; prove that the perimeter of the section is constant.

**171.** In a tetrahedron the sum of two opposite edges is equal to the sum of two other opposite edges. Prove that the sum of the dihedral angles whose edges are the first pair of lines is equal to the sum of the dihedral angles whose edges are the other pair of lines.

**172.**  $C'$  and  $D'$  are the feet of the perpendiculars drawn from any point to the faces opposite the vertices  $C, D$  of a tetrahedron  $ABCD$ .

Prove that  $\overline{AC'}^2 - \overline{BC'}^2 = \overline{AD'}^2 - \overline{BD'}^2$ .

**173.** If the opposite edges of a tetrahedron are perpendicular to each other, the perpendiculars drawn from the vertices to the opposite faces meet in a point.

**174.** If a tetrahedron is cut by a plane which passes through the middle points of two opposite edges, the section is divided into two equivalent triangles by the straight line joining these points.

**175.** From the middle point of one of the edges of a regular tetrahedron a fly descends by crawling around the tetrahedron, and reaches the base at the point where this edge meets the base. Find at what points the fly must cross the other edges if its path is everywhere equally inclined to the plane of the base.

**176.** The plane bisecting a dihedral angle of a tetrahedron divides the opposite edge into segments which are proportional to the faces which include the dihedral angle.

**177.** Straight lines are drawn from the vertices  $A, B, C, D$  of a tetrahedron through a point  $P$ , to meet the opposite faces in  $A', B', C', D'$ .

Prove that  $\frac{PA'}{AA'} + \frac{PB'}{BB'} + \frac{PC'}{CC'} + \frac{PD'}{DD'} = 1$ .

**178.** If  $a$  is the edge of a regular tetrahedron, its volume is  $\frac{a^3}{12}\sqrt{2}$ .

**179.** If  $a$  is the edge of a regular octahedron, its volume is  $\frac{a^3}{3}\sqrt{2}$ .

**180.** The lateral surface of a pyramid is greater than its base.

**181.** The volume of a triangular prism is equal to the area of a lateral face multiplied by one-half its perpendicular distance from any point in the opposite lateral edge.

**182.** The volume of a regular prism is equal to the product of its lateral area by one-half the apothem of its base.

**183.** The three lateral faces of a tetraedron are perpendicular to each other. If a triangle drawn in the base is projected on each of the three lateral faces, prove that the sum of the pyramids having these projections as bases and a common vertex anywhere in the base of the given tetraedron is equivalent to the pyramid having the given triangle for its base and its vertex at the vertex of the given tetraedron.

**184.** Extend the last exercise to the case where the common vertex is at any point in the plane of the base by regarding the volume of a pyramid as negative if the altitude is in the opposite direction from that in which it was measured for the pyramid on the same base in the last exercise.

**185. Defs.**—If  $ABCD$  is a rectangle, and  $EF$  a straight line parallel to  $AB$ , and not in the plane of the rectangle, the solid bounded by the rectangle  $ABCD$ , the trapezoids  $ABFE$ ,  $CDEF$ , and the triangles  $ADE$ ,  $BCF$  is a **wedge**.

The rectangle is called the **back** of the wedge; the trapezoids, its **faces**; the triangles, its **ends**; the line  $EF$ , its **edge**;  $AB$  is the length of the back and  $AD$  its breadth; the perpendicular from any point of  $EF$  upon the back is the **altitude** of the wedge.

**186.** If  $h$  is the altitude, prove that the volume of the above wedge is  $\frac{1}{6}h \times AD \times (2AB + EF)$ .

**187. Defs.**—If  $ABCD$  and  $EFGH$  are two rectangles lying in parallel planes,  $AB$  and  $BC$  being parallel to  $EF$  and  $FG$ , respectively, the solid bounded by these two rectangles and the trapezoids  $ABFE$ ,  $BCGF$ ,  $CDHG$ ,  $DAEH$ , is called a **rectangular prismoid**. The rectangles are called the **bases** of the prismoid and the perpendicular distance between them the **altitude**.

**188.** Prove that the volume of a rectangular prismoid is equal to the product of the sum of its bases, plus four times a section equidistant from the bases, multiplied by one-sixth the altitude.

#### PROBLEMS OF CONSTRUCTION

**189.** Having given the four perpendiculars from the vertices of a tetraedron to the opposite faces, and the distance of a point in space from three of the faces, find its distance from the fourth face.

**190.** Through a given straight line in one of the faces of a tetraedron pass a plane which shall cut off from the tetraedron another tetraedron which is to the first in a given ratio.

**191.** Find two straight lines whose ratio shall be the ratio of the volumes of two given cubes.

**192.** Find a point within a given tetraedron, such that the four pyramids having this point for vertex, and the faces of the tetraedron for bases, shall be equivalent.

#### PROBLEMS FOR COMPUTATION

**193.** (1.) Find the lateral area, total area, and volume of a regular triangular prism the perimeter of whose base is 16.413 in. and whose altitude is 14.718 in.

(2.) Find the lateral area, total area, and volume of a regular hexagonal pyramid each side of whose base is 8.84 in. and whose altitude is 4.92 in.

(3.) The area of the base of a pyramid is 13 sq. m.; its altitude is 4 m. Find the area of a section parallel to the base and distant  $1\frac{1}{2}$  m. from it. Also find the volume of the pyramid cut off by this plane.

(4.) Find the volume of a frustum of a pyramid whose base is a regular octagon having each side equal to 4 in., and whose altitude is 9 in., made by a plane 5 in. from the vertex.

(5.) The diagonal of a cube is 24.16 cm. Find its surface and volume.

(6.) The volume of a polyedron is 984.62 cu. ft. Find the volume of a similar polyedron whose edges are nine times the edges of the first polyedron.

(7.) The volume of a given tetraedron is 6.86 cu. m. Find the volume of the tetraedron whose vertices are a vertex of the given tetraedron and the intersections of the medians of the faces including that vertex.

(8.) Find the surface and volume of a regular tetraedron whose edge is 1.

(9.) Find the surface and volume of a regular octaedron whose edge is 16.247 mm.

(10.) Find the ratio of the volumes of a cube and a regular tetrahedron whose edges are equal.

(11.) Find the ratio of the volumes of a regular octahedron and a regular tetrahedron whose edges are equal.

(12.) Find the number of cubic feet of water that will be contained by a trench in the shape of a wedge the length of whose back is 20 m., whose breadth is 3 m., whose edge is 16 m., and whose depth is  $2\frac{1}{2}$  m. How many pounds of water will the trench hold, each cubic foot of water weighing  $62\frac{1}{2}$  lbs.? How many metric tons?

(13.) An embankment is in the form of a rectangular prismoid. The length and breadth of its base are 246 ft. and 8 ft.; the length and breadth of its top are 239 ft. and 3 ft. Its height is 4 ft. Find the number of cubic yards of earth it contains.

## BOOK VIII

### PROBLEMS OF DEMONSTRATION

**194.** If two circles in space are such that their centres are the projections of the same point on their planes, and the tangents to the circles drawn from a point in the intersection of their planes are equal, the two circles are on the same sphere.

**195.** If through a fixed point within or without a sphere three straight lines are drawn at right angles to each other so as to intersect the surface of the sphere, the sum of the squares of the three chords thus formed is constant. Also the sum of the squares of the six segments of these chords is constant.

**196.** If three radii of any sphere perpendicular to each other are projected upon any plane, the sum of the squares of the three projections is equal to twice the square of the radius of the sphere.

**197.** If from a point without a sphere any number of straight lines be drawn to touch the sphere, the points of contact will all be in one plane.

**198.** A sphere can be inscribed in or circumscribed about any regular polyhedron.

**199.** A regular tetraedron and a regular octaedron are inscribed in the same sphere; compare the radii of the spheres which can be inscribed in the tetraedron and in the octaedron.

**200.** From any point  $P$  in a diameter of a given sphere straight lines  $PQ, PR$  are drawn perpendicular to that diameter, in any direction and of any length, provided  $Q$  and  $R$  lie within the sphere. Through  $P, Q, R$  two spherical surfaces are passed touching the given spherical surface. Prove that the sum of their radii is equal to the radius of the given sphere.

**201.** If a square is inscribed in a face of a cube, the plane determined by one side of the square and the corner of the opposite face in the same edge as the adjacent corner of the same face, touches the inscribed sphere.

**202.** If the opposite edges of a tetraedron are equal, its four vertices may be taken as four non-adjacent vertices of a rectangular parallelepiped. Prove that of the five spheres touching the faces of such a tetraedron, or the faces produced, four have their centres at the remaining four vertices of the parallelepiped, and the fifth at the intersection of the diagonals of the parallelepiped.

**203.**  $ABC$  is a spherical triangle and  $AT$  an arc of a great circle tangent to the circumscribing small circle at  $A$ . Prove that the angle  $BAT$  is equal to the angle  $C$  minus half the spherical excess of the triangle.

**204.**  $ABC$  is a spherical triangle; through the middle points of  $AB$  and  $AC$  an arc of a great circle is drawn cutting  $BC$  produced in  $D$ . Prove that  $DB$  is the supplement of  $DC$ .

**205.** How many spheres, each equal to a given sphere, can be tangent to the given sphere at the same time?

#### LOCI

**206.** Find the locus of a point whose distances from three given points in space are in the ratio of three given lines.

**207.** Find the locus of the intersection of planes tangent to a sphere at the extremities of chords which pass through a fixed point.

**208.** Find the locus of a point which divides in a given ratio a straight line drawn from a fixed point to the surface of a sphere.

## PROBLEMS OF CONSTRUCTION

**209.** Find the centre of a sphere which passes through the circumference of a given circle, and through a given point not in the plane of the circle.

**210.** Through a given point pass two spherical surfaces tangent to a given sphere.

**211.** Find the radius of a sphere which shall circumscribe four equal spheres which touch each other.

## PROBLEMS FOR COMPUTATION

**212.** (1.) The area of the base of a circular cone is 43 sq. in. Its altitude is 19 in. Find the area of a section parallel to the base and 10 in. from the vertex.

(2.) If the area of a circle of a sphere distant 10 cm. from its centre is 40 sq. cm., find the radius of the sphere.

(3.) The polar distance of a circle of a sphere is  $30^\circ$ . If its circumference is 6 m., what is the radius of the sphere?

(4.) Find the area in square feet of a lune whose angle is  $36^\circ$  on a sphere whose surface is 46 sq. m.

(5.) If the area of a spherical triangle whose angles are  $110^\circ$ ,  $46^\circ$ , and  $150^\circ$ , is 84.662 sq. yds., what is the area of a trirectangular triangle on the same sphere?

(6.) The angles of a spherical pentagon are  $68^\circ$ ,  $97^\circ$ ,  $156^\circ$ ,  $80^\circ$ , and  $142^\circ$ . Its area is 8 sq. ft. Find the area of the sphere.

(7.) Find the volume of a spherical ungula the angles of whose base are each  $43^\circ$ , in a sphere whose volume is 18,561 cu. m.

(8.) Find the volume of a spherical pyramid the angles of whose base are  $70^\circ$ ,  $98^\circ$ ,  $153^\circ$ ,  $89^\circ$ , and  $150^\circ$ , in a sphere whose volume is 77.253 cu. yds.

(9.) The radii of two spheres are 14 m. and 9 m. respectively. The distance between their centres is 20 m. Find the length of the circumference in which their surfaces intersect.



(10.) Find the radii of the spheres inscribed in and circumscribed about a regular tetraedron whose edge is 6.5438 in.

(11.) If the radius of the sphere circumscribed about a cube is 10.643 ft., find the volume of the cube.

(12.) Find the surface of a regular octaedron, the radius of whose circumscribed sphere is 32.147 in.

(13.) A hollow cone of revolution of which the altitude is equal to three-fourths the slant height is cut open in a straight line drawn from the vertex to a point in the base. Find (in right angles) the vertical angle of the unrolled surface.

## BOOK IX

### PROBLEMS OF DEMONSTRATION

**213.** The volume of a cylinder of revolution is equal to its lateral area multiplied by one-half the radius of its base.

**214.** The volume of a cylinder of revolution is equal to the area of its generating rectangle multiplied by the circumference of a circle whose radius is the distance from the centre of the rectangle to the axis.

**215.** The volume of a cone of revolution is equal to the area of its generating triangle multiplied by the circumference of a circle whose radius is the distance from the intersection of the medians of the triangle to the axis.

**216.** Express the volume of a cone of revolution in terms of its lateral area and the perpendicular from the centre of its base upon an element.

**217.** Express the volume of a cone of revolution in terms of its total surface and the radius of the inscribed sphere.

**218.** The volumes of polyedrons circumscribed about equal spheres are proportional to their surfaces.

**219.** Two spheres intersect, the centre of the first lying on the surface of the second. Prove that the surface intercepted by the first on the second is independent of the size of the second sphere. Prove that this surface is one-fourth the surface of the first sphere.

## PROBLEMS OF CONSTRUCTION

**220.** Cut a sphere by a plane so that the area of the section shall be equal to the difference of the areas of the two zones which the plane determines.

**221.** Divide a zone in mean and extreme ratio by a plane parallel to its bases.

**222.** Inscribe in and also circumscribe about a sphere a cone of which the total area shall be in a given ratio to the area of the sphere.

**223.** Inscribe in and also circumscribe about a given sphere a cone of which the volume shall be in a given ratio to the volume of the sphere.

**224.** Determine a point on the diameter of a sphere such that if a plane is passed through this point perpendicular to the diameter, the surface of the zone limited by this plane and containing the nearer extremity of the diameter shall be equal to the lateral surface of the cone whose base is the circle of intersection of the plane with the sphere and whose vertex is the farther extremity of the diameter.

## PROBLEMS FOR COMPUTATION

**225.** (1.) If the perimeter of a right section of a cylinder is 16 in., and its lateral area is 256 sq. in., what is the length of an element?

(2.) Find the volume of a cylinder of revolution whose total area is  $160\pi$  and whose radius is 4.

(3.) Find the radius of a cylinder of revolution whose total area is  $80\pi$  and whose altitude is 6.

(4.) An oil tank is in the form of a circular cylinder. If the tank is 26 ft. long and 78 in. in diameter, how many liters of oil will it contain?

(5.) Find the volume of a cone of revolution whose total area is  $200\pi$  and whose altitude is 16.

(6.) The lateral area of a cone of revolution is  $39\pi$ . Its altitude is 9. Find the height of an equivalent cylinder of revolution whose radius is 4.

(7.) In a sphere whose diameter is 14 in. the altitude of a zone of

one base is 2 in. Find the altitude of a cylinder of revolution whose lateral area shall equal the area of the zone and whose base shall equal the base of the zone.

(8.) Find the radius of a circle whose area shall equal the area of a zone of altitude 16.954 m. on a sphere whose diameter is 20 m.

(9.) Find the radius of a sphere whose area shall equal the area of the zone in the previous example.

(10.) A conical glass is 5 in. high and 4 in. across at the top. A marble is within the glass and water is poured in till the marble is just immersed. If the amount of water poured in is  $\frac{1}{4}$  the contents of the glass, what is the diameter of the marble?

(11.) If two spheres of radii 13 in. and 8 in. are inscribed in a cone of revolution so that the greater may touch the less and also the base of the cone, find the volume of the cone.

(12.) A sphere and an octaedron are inscribed in the same cube, the vertices of the octaedron being at the centres of the faces of the cube. Compare the volumes of the three solids.

(13.) Find the ratio of the volume of a sphere touching the edges of a regular tetraedron to the volume of a sphere touching one face and the other faces produced.

(14.) The volume of a spherical sector is 19.463 cu. mm. Its base is one-third the surface of the sphere. Find the surface of the sphere.

(15.) Find the volume of a spherical shell whose two surfaces are  $20\pi$  and  $15\pi$ .

(16.) Find the volume of a spherical segment whose altitude is 9 in. and the radii of whose bases are 4 in. and 5 in.

(17.) Assuming the diameter of the earth to be 7960 miles, what is the area of the portion which would be visible from a point 3980 miles above its surface?

(18.) Show that if  $R$  is the radius of the earth and  $h$  the height of a point of observation above its surface, the area of the visible surface

$$\text{is } \frac{2\pi R^2 h}{R+h}.$$

(19.) In a sphere whose radius is 20 in. find the volume of a segment of one base whose altitude is 6 in.

(20.) Show that if  $R$  is the radius of a sphere and  $h$  the altitude of a spherical segment of one base, the volume of the spherical segment is  $\pi h^2(R - \frac{1}{3}h)$ .

PROBLEMS IN MAXIMA AND MINIMA IN PLANE AND SOLID GEOMETRY

**226.** Through a given point draw a straight line which shall form with two given lines a triangle of minimum area.

**227.** Through a given point within a given angle draw a straight line which shall form with the sides of the given angle a triangle of minimum perimeter.

**228.** Through the intersection of two tangents to a circle draw a straight line cutting the circumference in two points such that, if they are joined to the points of tangency, the product of either pair of opposite sides of the inscribed quadrilateral thus formed shall be a maximum.

**229.** In an acute-angled triangle inscribe a rectangle, such that its diagonal shall be a minimum.

**230.** From a given point in a diameter  $AB$  of a circle produced draw a straight line cutting the circumference in two points  $C$  and  $H$ , so that the triangle  $ACH$  shall be a maximum.

**231.** Two straight lines containing a given angle are drawn from a given point in the base of a triangle, forming a quadrilateral with the two other sides. Prove that, of all the quadrilaterals which may be thus formed, that one whose sides passing through the given point are equal is a maximum, if the given angle is less than the supplement of the opposite angle of the triangle; a minimum, if the given angle is greater than the supplement of the opposite angle; neither a maximum nor a minimum, if the given angle is equal to the supplement of the opposite angle.

**232.** In the last exercise a maximum or a minimum quadrilateral can be formed for each point in the base (except in the case when the given angle is the supplement of the opposite angle of the triangle). Prove that, of all these maximum or minimum quadrilaterals, the least maximum or the greatest minimum is that whose equal sides make equal angles with the base.

**233.** Find a point in a plane such that the sum of its distances from two fixed points on the same side of the plane shall be a minimum.

**234.** Find a point in a plane such that the difference of its distances from two fixed points on opposite sides of the plane shall be a maximum.

**235.** Of all quadrangular prisms of which the volumes are equal, the cube has the least surface.

# INTRODUCTION TO MODERN GEOMETRY

[The numbers of the figures are the same as of the articles to which they belong.]

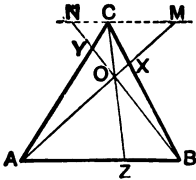
## DIVISION OF LINES. THE COMPLETE QUADRILATERAL

**1.** The lines connecting any point with the three vertices of a triangle so divide the opposite sides that the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments.

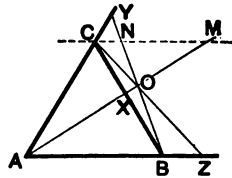
*Hint.*—Draw  $MCN$  parallel to  $AB$ .

From similar triangles,

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{CM}{CN} \cdot \frac{AB}{CM} \cdot \frac{CN}{AB} = 1.$$



FIGS. 1 AND 2



FIGS. 1 AND 2

**2.** Conversely, if the sides of a triangle are so divided (either two or not any of the points of division being on the sides produced) that the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments, the lines connecting the points of division with the opposite vertices meet in a point.

*Hint.*—Use the method of reductio ad absurdum.

**3. Def.**—A line which cuts a system of lines is a **transversal**.

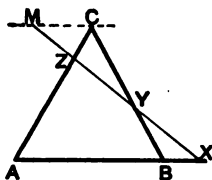
In § 4 and § 5  $XZ$  is a transversal which cuts the lines  $AB$ ,  $AC$ ,  $BC$ .

**4.** If the sides of a triangle are cut by a transversal, the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments.

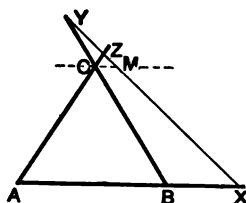
*Hint.*—Draw  $CM$  parallel to  $AB$ .

From similar triangles,

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = \frac{AX}{XB} \cdot \frac{XB}{CM} \cdot \frac{CM}{AX} = 1.$$



FIGS. 4 AND 5



FIGS. 4 AND 5

**5.** Conversely, if the sides of a triangle are so divided (either one or three of the points of division being on the sides produced) that the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments, the points of division are in a straight line.

*Hint.*—Use the method of reductio ad absurdum.

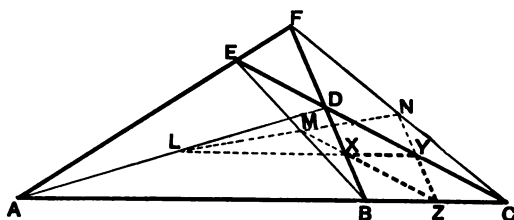
**6. Exercise.**—If  $ABCD$  be four points taken in order on a straight line,  
 $AB \cdot CD + BC \cdot AD = AC \cdot BD$ .



FIG. 6

**7. Def.**—A **complete quadrilateral** is the figure formed by four straight lines intersecting in six points. The six points are the vertices; the three lines connecting opposite vertices are the diagonals.

$ABCDEF$  is a complete quadrilateral;  $AD$ ,  $BE$ ,  $CF$ , are the diagonals.



FIGS. 7 AND 8

8. The middle points of the diagonals of a complete quadrilateral are in a straight line.

*Hint.*—Let  $L, M, N$  be the middle points of the diagonals. Construct the triangle  $XYZ$ , whose vertices are at the middle points of  $BD, DC, CB$ ; the sides of this triangle pass through  $L, M, N$ .

Since  $FA$  is a transversal cutting the sides of the triangle  $BCD$ ,

$$DF \cdot BA \cdot CE = FB \cdot AC \cdot ED. \quad \S 4$$

But  $YN = \frac{1}{2}DF, NZ = \frac{1}{2}FB$ , etc.

Hence  $YN \cdot XL \cdot ZM = NZ \cdot LY \cdot MX$ .

Therefore the points  $L, M, N$ , being on the sides of the triangle  $XYZ$ , are in a straight line. § 5

HARMONIC SECTION

9. *Def.*—If a line  $AB$  is divided harmonically at  $C$  and  $D$ , the points  $C$  and  $D$  are **harmonic conjugates** to the points  $A$  and  $B$ . The four points  $A, B, C, D$  are **harmonic points**, and  $AB$  is a **harmonic mean** between  $AC$  and  $AD$ .

A line is divided harmonically if it is divided internally and externally in the same ratio. § 332, p. 151

Thus, if 
$$\frac{AC}{CB} = \frac{AD}{DB},$$

$AB$  is divided harmonically at  $C$  and  $D$ .

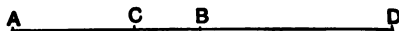


FIG. 9

10. *Exercise.*—The above definition of a harmonic mean is equivalent to the algebraic definition.

*Hint.*—In Algebra the harmonic mean between  $a$  and  $b$  is  $\frac{2ab}{a+b}$ .



**11. Def.**—A pencil of rays is a system of straight lines (rays) passing through a point (the vertex).

Thus  $OA, OB, OC, OD$ , Fig. 13, form a pencil of rays.

**12. Def.**—A harmonic pencil is a pencil of four rays which pass through the harmonic points of a line.

**13.** Any transversal is cut harmonically by a harmonic pencil.

*Hint.*—Let  $A, B, C, D$  be harmonic points,  $P$  and  $p$  the perpendiculars from  $O$  on  $AD$  and  $ad$ . To prove  $\frac{ac}{cb} = \frac{ad}{db}$ , that is,  $\frac{ac \cdot db}{ad \cdot cb} = 1$ .

The ratio of the areas of two corresponding triangles as  $aOc$  and  $AOC$  is

$$\frac{Oa \cdot Oc}{OA \cdot OC} = \frac{\frac{1}{2}p \cdot ac}{\frac{1}{2}P \cdot AC}. \quad \text{§ 398, p. 180}$$

$$\text{Hence } \frac{ac \cdot db}{ad \cdot cb} = \frac{Oa \cdot Oc \cdot Od \cdot Ob}{OA \cdot OC \cdot OD \cdot OB} \times \frac{AC \cdot DB}{AD \cdot CB} = \frac{AC \cdot DB}{AD \cdot CB} = 1.$$

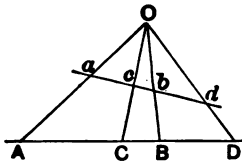


FIG. 13

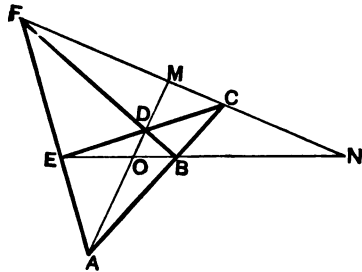


FIG. 14

**14.** Each diagonal of a complete quadrilateral is divided harmonically by the other two.

*Hint.*—Since  $BN$  is a transversal cutting the sides of the triangle  $ACF$ ,

$$(1) \quad AB \cdot CN \cdot FE = BC \cdot NF \cdot EA. \quad \text{§ 4}$$

$$(2) \quad \text{Also } AB \cdot CM \cdot FE = BC \cdot MF \cdot EA. \quad \text{§ 1}$$

$$\text{By dividing (1) by (2)} \quad \frac{CN}{CM} = \frac{NF}{MF}.$$

$$\text{Therefore} \quad \frac{CM}{MF} = \frac{CN}{NF}.$$

15. If two harmonic pencils have one pair of corresponding rays coincident, the intersections of the other three pairs of corresponding rays are in a straight line.

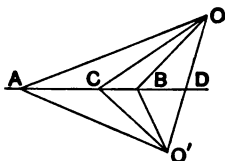


FIG. 15

*Hint.*—Use the method of reductio ad absurdum.

SOME PROPERTIES OF CIRCLES

16. The product of the perpendiculars drawn from a point on a circle\* to two tangents is equal to the square of the perpendicular drawn from the point to their chord of contact.

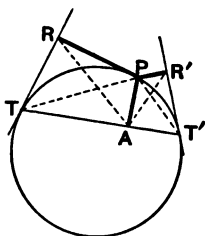


FIG. 16

*Hint.*—Let  $RT, R'T'$  be the tangents and  $TT'$  their chord of contact.

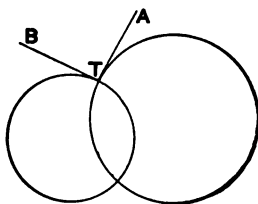
A circle can be circumscribed about each of the quadrilaterals  $APRT$  and  $APR'T'$ , since the sum of the opposite angles in each is equal to two right angles.

Hence angle  $ARP = ATP = PT'R' = R'AP$ . Likewise angle  $AR'P = PAR$ .

Therefore the triangles  $ARP$  and  $R'AP$  are similar and  $AP^2 = PR \cdot PR'$ .

\* The word circle instead of circumference is used except where ambiguity would result.

**17. Def.**—The angle at which two circles cut each other is the angle between the tangents drawn at the point of intersection.



FIGS. 17 AND 18

**18. Def.**—If two circles cut each other at right angles they are said to cut **orthogonally**.

**19.** If the square of the distance between the centres of two circles is equal to the sum of the squares of their radii, the circles cut each other orthogonally and conversely.

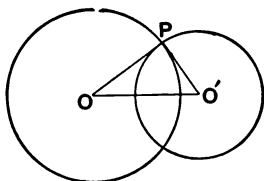


FIG. 19

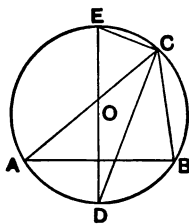


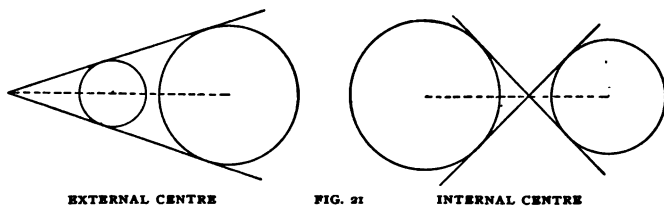
FIG. 20

**20.** If a circle be circumscribed about a triangle, the lines joining the extremities of the diameter which is perpendicular to the base, to the vertex, are the internal and external bisectors of the vertex angle.

*Hint.*—Angle  $DCE$  is a right angle.

**21. Defs.**—The point of intersection of the direct common tangents of two circles is their **external centre of similitude**; the point of intersection of their inverse common tangents, their **internal centre of similitude**.

The centres of similitude are on the line of centres of the circles, and divide that line externally and internally in the ratio of the radii.



**22.** The six centres of similitude of three circles are three by three on four straight lines.

The three external centres of similitude are in a straight line, and each pair of internal centres of similitude is in a straight line passing through an external centre of similitude.

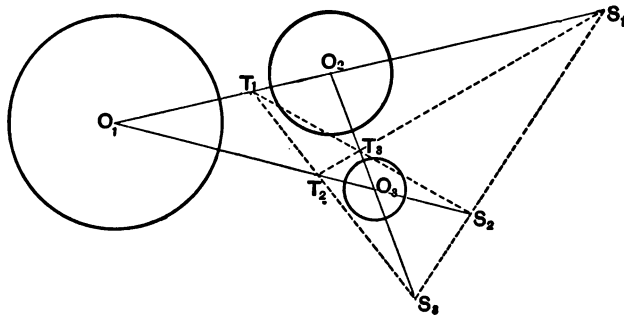


FIG. 22

*Hint.*—Let  $S_1, S_2, S_3$  be the external,  $T_1, T_2, T_3$  the internal centres of similitude ; also let  $R_1, R_2, R_3$  be the radii of the circles.

$$\frac{S_1O_1}{S_1O_2} \cdot \frac{S_2O_2}{S_2O_1} \cdot \frac{S_3O_3}{S_3O_1} = \frac{R_1}{R_2} \cdot \frac{R_2}{R_1} \cdot \frac{R_3}{R_3} = 1.$$

Hence  $S_1, S_2, S_3$  are in a straight line. § 5

Again 
$$\frac{S_2O_2}{S_2O_3} \cdot \frac{T_1O_1}{T_1O_2} \cdot \frac{T_3O_3}{T_3O_1} = \frac{R_2}{R_3} \cdot \frac{R_1}{R_2} \cdot \frac{R_3}{R_1} = 1.$$

Hence  $T_1, T_2, T_3$  are in a straight line. § 5

**23. Cor.**—If a variable circle touch two fixed circles, the chord of contact passes through an external centre of similitude of the fixed circles ; for each point of contact is a centre of similitude of the variable circle and one of the fixed circles.

## INVERSION

**24. Def.**—Two points are inverse to each other with respect to a given centre of inversion if they are in the same straight line with this centre, and if the product of their distances from it is equal to a constant.

Two curves are inverse to each other if the successive points of the one invert into the successive points of the other with respect to a given centre.



FIG. 24

$Q$  is the inverse of  $P$  with respect to the centre,  $O$  if  $OP = \frac{K}{OQ}$ .

The curve  $Y$  is the inverse of the curve  $X$ , if, for every point  $P$  of  $X$  there is a point  $Q$  of  $Y$  such that  $OP \cdot OQ = K$ .

**25.** The inverse of a circle is a straight line if the centre of inversion is on the circle.

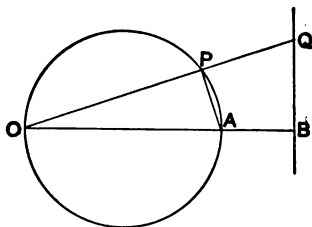


FIG. 25

*Hint.*— $OP \cdot OQ = OA \cdot OB = K$ .

26. This principle makes it possible to draw a line mathematically straight; for in the four linkages\* shown below the point  $P$  inverts into  $Q$  with respect to the centre  $O$ , and if  $P$  move in a circle passing through the fixed point  $O$ , then  $Q$  will move in a straight line.

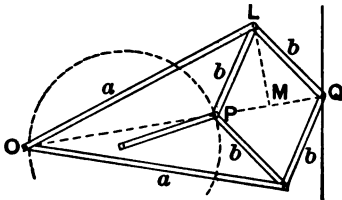


FIG. 26 (1)

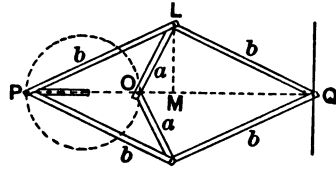


FIG. 26 (2)

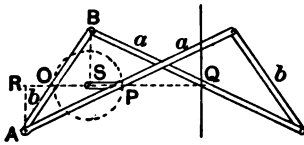


FIG. 26 (3)

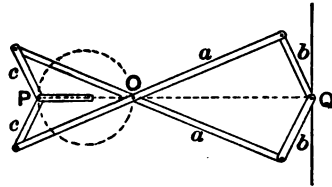


FIG. 26 (4)

In each linkage the bars (links) denoted by the same letter are equal. To prove the property of inversion:

In linkages (1) and (2)

$$OP = OM - PM, \text{ and } OQ = OM + PM; \text{ then } OP \cdot OQ = OM^2 - PM^2.$$

$$\text{But } OM^2 = a^2 - LM^2 \text{ and } PM^2 = b^2 - LM^2.$$

$$\text{Therefore } OP \cdot OQ = a^2 - b^2, \text{ a constant.}$$

In linkage (3) the points  $O, P, Q$  divide the links in the same ratio.

$$OP = RP - RO = \sqrt{AP^2 - AR^2} - \sqrt{AO^2 - AR^2},$$

$$\text{and } OQ = OS + SQ = \sqrt{BO^2 - BS^2} + \sqrt{BQ^2 - BS^2}.$$

$$\text{But } BS = \frac{BO}{AO} AR, \text{ } BQ = \frac{BO}{AO} \cdot AP.$$

$$\text{Therefore } OP \cdot OQ = \frac{BO}{AO} \left\{ AP^2 - AO^2 \right\}, \text{ a constant.}$$

Compare linkage (4) with the pantograph.

\* A linkage is a system of bars pivoted together.

The original account of linkages (1) and (2) was published in "Nouvelles Annales," 1873; of (3) in the "Report of the British Association," 1874; of (4) in the "Report of the British Association," 1884.

**27.** The inverse of a circle is a second circle if the centre of inversion is not on the first circle.

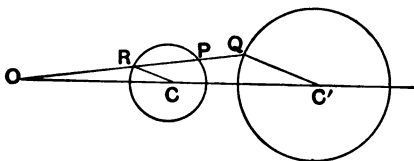


FIG. 27

*Hint.*—Let  $C$  be the first circle,  $O$  the centre of inversion,  $Q$  the inverse of  $P$ . Draw  $QC'$  parallel to  $RC$  to meet  $OC$  in  $C'$ .

Since  $OP \cdot OR$  is constant and  $OP \cdot OQ$  is constant,  $\frac{OQ}{OR}$  is constant.

By similar triangles  $\frac{OC'}{OC} = \frac{OQ}{OR}$  and  $\frac{C'Q}{CR} = \frac{OQ}{OR}$ .

Therefore  $C'$  is a fixed point and  $C'Q$  a constant length.

**28. Exercise.**—The centre of inversion is a centre of similitude of a given circle and its inverse.

**29. Exercise.**—If two circles touch each other, their inverses also touch each other.

**30. Exercise.**—A circle can be inverted into itself.

*Hint.*—The constant of inversion must be equal to the square of the tangent drawn to the circle from the centre of inversion.

**31.** The inverse of a sphere is a plane, if the centre of inversion is on the sphere.

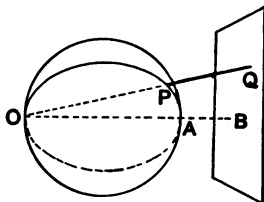


FIG. 31

*Hint.*—Every point on the sphere is on a great circle passing through the centre of inversion, and will invert into a point of the plane; compare with § 25.

**32.** The inverse of a sphere is a second sphere, if the centre of inversion is not on the first sphere.

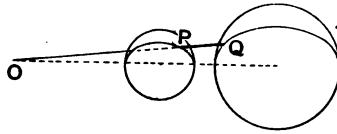


FIG. 32

*Hint.*—Compare with § 27.

**33.** If two circles intersect, their angle of intersection is equal to the angle of intersection of their inverses.

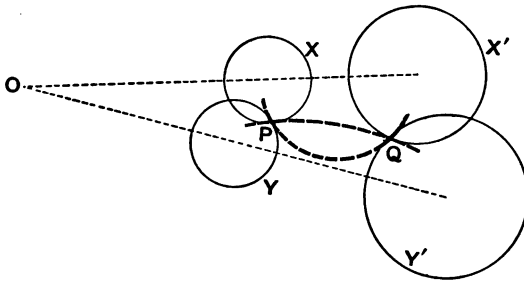


FIG. 33

*Hint.*—The circles  $X$  and  $Y$  invert into  $X'$  and  $Y'$ .

A circle can be described tangent to  $X$  at  $P$  and passing through  $Q$ . This circle inverts into itself and is therefore tangent to  $X'$  at  $Q$ . §§ 30, 29

Likewise a circle can be described tangent to  $Y$  at  $P$  and passing through  $Q$ . This circle is tangent to  $Y'$  at  $Q$ .

The angle at which these tangent circles intersect is equal to the angle at which  $X$  and  $Y$  intersect and also to the angle at which  $X'$  and  $Y'$  intersect.

**34. Cor.**—If a straight line and a circle, or two straight lines, intersect, their angle of intersection is equal to the angle of intersection of their inverses.

**35.** A single inversion may be found equivalent to any series of an odd number of inversions from the same centre.

*Hint.*—If  $a$  invert into  $b$ ,  $b$  into  $c$ ,  $c$  into  $d$ , . . .  $m$  into  $n$  where the number of inversions is odd, find an inversion by which  $a$  inverts into  $n$ .

Why does not this theorem apply to an even number of inversions?



## RADICAL AXIS AND COAXIAL CIRCLES

**36.** The locus of a point, from which tangents drawn to two circles are equal, is a straight line perpendicular to the line of centres.

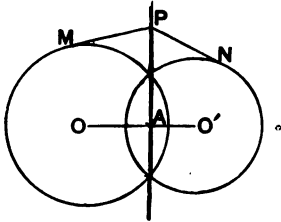


FIG. 36 (1)

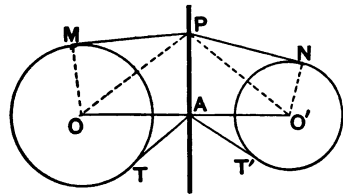


FIG. 36 (2)

*Hint.*—(1.) If the circles intersect, the locus is the common chord.

(2.) If the circles do not intersect, let  $A$  be the point in the line of centres from which tangents to the circles are equal. Erect the perpendicular  $AP$ .

$$\begin{aligned} PM^2 &= PO^2 - OM^2 \\ &= AO^2 + AP^2 - OM^2 \\ &= AT^2 + OM^2 + AP^2 - OM^2 = AT^2 + AP^2. \end{aligned}$$

Similarly  $PN^2 = AP^2 + AT'^2$ .

Therefore  $PM^2 = PN^2$ .

**37. Def.**—The straight line, which is the locus of the points from which tangents drawn to two circles are equal, is the **radical axis** of the circles.

**38.** The three radical axes of three circles meet in a point.

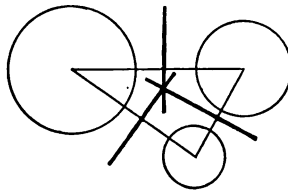


FIG. 38

*Hint.*—The tangents drawn to the three circles from the point of intersection of two of the radical axes are equal; hence the third radical axis must pass through the point.

**39.** The difference between the squares of the tangents drawn from any point to two circles is equal to twice the product of the distance of the point from the radical axis by the distance between the centres of the circles.

*Hint.*—Let  $C$  be the centre of  $OO'$ ,  $AB$  the radical axis,  $PR$  the perpendicular from  $P$  on  $OO'$ .

$$PT^2 - P'T'^2 = (PO^2 - OT^2) - (PO'^2 - O'T'^2).$$

$$PO^2 - PO'^2 = OR^2 - O'R^2 = 2OO'.CR.$$

$$OT^2 - O'T'^2 = OA^2 - O'A^2 = 2OO'.AC.$$

Therefore

$$PT^2 - P'T'^2 = 2OO'.AR.$$

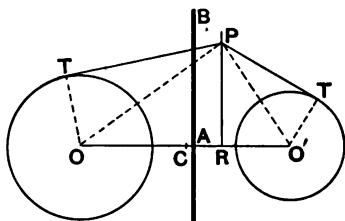


FIG. 39

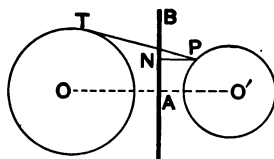


FIG. 40

**40. Cor.**—The square of the tangent drawn from a point on one circle to another circle is equal to twice the product of the distance between the centres of the circles by the distance of the point from their radical axis.

**41. Def.**—A system of circles such that some line is a radical axis common to every pair of circles of the system is a **coaxal system**.

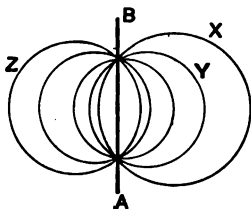


FIG. 41 (1)

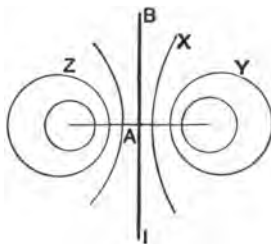


FIG. 41 (2)

Thus if  $AB$  is the radical axis of the circles  $X$  and  $Y$ ,  $X$  and  $Z$ ,  $Y$  and  $Z$ , etc., the system of circles is coaxal.

**42.** To describe a system of circles coaxal with two given circles.

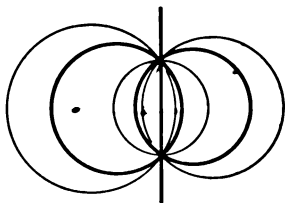


FIG. 42 (1)

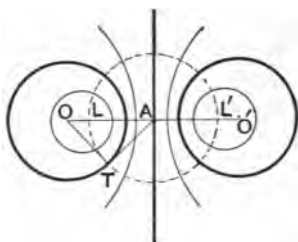


FIG. 42 (2)

(1.) If the circles intersect.

*Hint.*—The common chord is the radical axis, and any circle through the points of intersection of the two circles is coaxal with them.

(2.) If the circles do not intersect.

*Hint.*—Let  $A$  be the intersection of the radical axis with the line of centres. About  $A$  with a radius equal to the tangent  $AT$ , describe a circle. This circle will cut the given circles orthogonally. Any circle which has its centre on  $OO'$  and is cut orthogonally by this circle is coaxal with the given circles. §§ 37, 41

In a system of coaxal circles which do not intersect, the circles grow smaller as the points in which they are cut by the orthogonal circle approach more nearly the line of centres. The points in which the orthogonal circle cuts the line of centres may be considered as circles of indefinitely small radius, the limiting circles of the system.

*Def.*—These points are called the **limiting points** of the system.

Thus in Fig. 42 (2) the limiting points are  $L, L'$ .

**43.** From the last article it follows that there are two forms of coaxal circles; in the one the circles intersect and there are no limiting points, in the other the circles do not intersect and there are limiting points.

**44.** The following special cases of the theorem of § 39 are of importance.

(1.) The square of the distance from any point  $P$  of a given circle of a coaxal system to either of the limiting points of the system is proportional to the distance of  $P$  from the radical axis.

(2.) If three circles are coaxial, the tangents drawn from any point of the first to the other two are in a given ratio.

(3.) If tangents drawn from a variable point to two given circles are in a given ratio, the locus of the point is a circle coaxial with the given circles.

**45.** A system of coaxial circles can be described such that each circle will cut orthogonally all the circles of a given coaxial system.

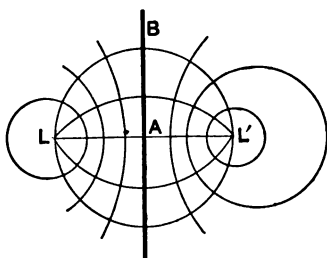


FIG. 45

*Hint.*—The limiting points of the one system will be the points of intersection of the circles of the other.

*Def.*—The two systems are called **orthogonal systems** of coaxial circles.

**46. Exercise.**—If a system of circles is cut orthogonally by two circles it is a coaxial system.

**47.** The inverse of a system of concentric circles is a coaxial system.

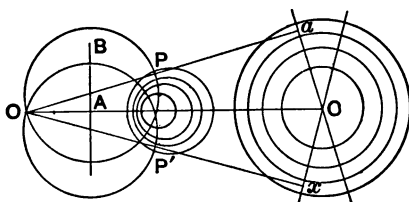


FIG. 47

*Hint.*—Two straight lines through the centre of the concentric system will invert into two circles cutting the circles of the inverse of the concentric system orthogonally.

**48. Remark.**—A system of straight lines passing through a point is a system of intersecting coaxial circles; the other point of intersection is at infinity. A system of concentric circles is a system of non-intersecting coaxial circles; the centre is one of the limiting points, the other limiting point is at infinity.

**49.** Lines drawn through either of the points of intersection of a system of intersecting coaxial circles are divided proportionally by the circles.

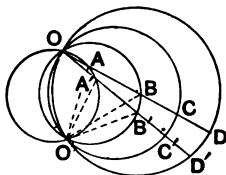


FIG. 48

*Hint.*—To prove  $\frac{AB}{A'B'} = \frac{BC}{B'C'}$  = etc.

Angle  $OA'O' = OA'O'$ ; angle  $OBO' = OB'O'$ .

§ 201, p. 96

Hence triangle  $AO'B$  is similar to triangle  $A'O'B'$ .

### THE STEREOGRAPHIC PROJECTION

**50.** The stereographic projection furnishes a useful and interesting application of the principles of inversion and coaxial circles.

It has been shown in § 31 that if a sphere be inverted from a point on itself, the inverse is a plane. The result of such an inversion is

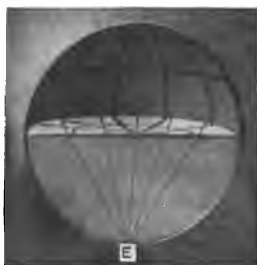


FIG. 50 (a)



FIG. 50 (b)

the **stereographic projection** of the sphere. In this projection any figure on the sphere is represented by the figure on the plane into which it inverts; in the inversion, angles of the figure on the sphere and the corresponding angles of its projection on the plane are equal.

The stereographic projection may also be defined as follows: Suppose a transparent sphere have opaque meridians, parallels of latitude, and other lines or figures drawn upon it. The stereographic projection is the picture of these lines and figures obtained if a photographic lens have its optical centre on the surface of the sphere. Or, it is the shadow cast upon a plane without the sphere, if a point of light be at the farther extremity of the diameter perpendicular to this plane. Again, if a line be drawn from the extremity of a diameter of the sphere to any point on the surface of the sphere, its intersection with a plane perpendicular to the diameter is the stereographic projection of this point.

Thus Figures (1), (2), (3), show three forms of the stereographic projection upon a diametral plane. Any plane parallel to this diametral plane would serve as well, and the figures upon the two planes would be similar. The centre of inversion is at  $E$  in each case.

*Exercise.*—Prove by aid of the triangles  $PRS$ ,  $QRS$  of Fig. (4), that the assumption that angles are preserved in this projection is correct. Prove also from Fig. (4) that a circle projects into a circle.

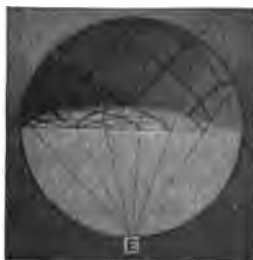


FIG. 50 (3)

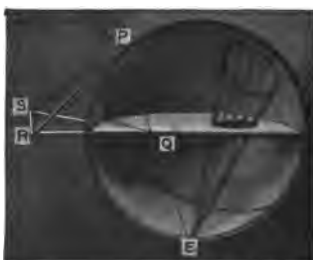


FIG. 50 (4)

The **equatorial stereographic projection** is that obtained if the centre of inversion is at one of the poles of the sphere; it is shown in Fig. (1) and Fig. (5). The parallels of latitude are represented by concentric circles of which the centre is the opposite pole, and the meridians by straight lines through this centre.

The **meridional stereographic projection** is that obtained if the centre of inversion is on the equator; it is shown in Fig. (2) and Fig. (6). The parallels of latitude are represented by a system of coaxial circles of which the poles are the limiting points, and the equator the radical axis. The meridians are represented by a system of intersecting coaxial circles of which the poles are the points of intersection.

The **horizontal stereographic projection** is that obtained if the centre of inversion is on a parallel of latitude other than the equator. It is shown in Fig. (3) and Fig. (7). The parallels of latitude invert into a system of non-intersecting coaxial circles; the poles inverting into the limiting points, and the parallel through the centre of inversion into the radical axis. The meridians invert into a system of intersecting coaxial circles, the poles inverting into their points of intersection.

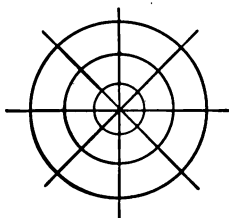


FIG. 50 (5)

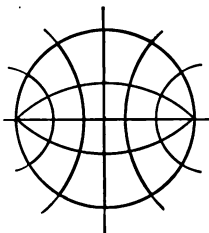


FIG. 50 (6)

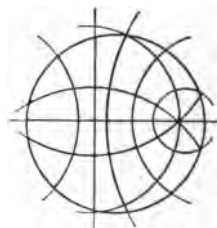


FIG. 50 (7)

It has already been shown that a system of concentric circles and a system of straight lines passing through their centre invert into orthogonal systems of coaxial circles. In accordance with this principle the equatorial projection can be inverted into either the meridional or any desired horizontal projection by properly choosing the centre of inversion; pictures of this inversion, as performed by one of the linkages before described, are shown in Figs. (8) and (9).<sup>\*</sup> Moreover, the meridional projection can be inverted into any desired form of the horizontal in the same manner. It is possible then by a proper choice of the centre of inversion to invert any form of the stereographic projection into any other form desired.

<sup>\*</sup> "Report of the British Association," 1884.

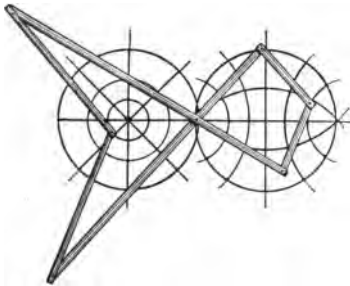


FIG. 50 (8)

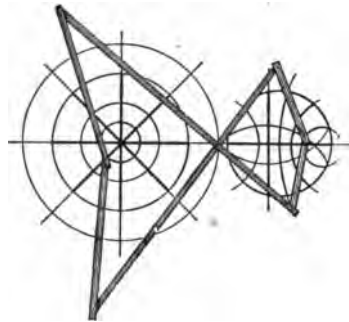


FIG. 50 (9)

POLES AND POLARS

**51. Def.**—If a point is taken on the radius of a circle and another point on the same radius produced, so that the product of their distances from the centre is equal to the square of the radius, each is the pole of the line (its polar) drawn through the other perpendicular to the radius.

Thus if  $OP.OQ = R^2$  the point  $P$  is the pole of the line  $QS$ , and the line  $QS$  is the polar of the point  $P$  with respect to the circle  $X$ .

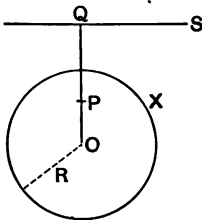


FIG. 51

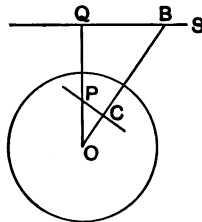


FIG. 52

**52.** If a line passes through a given point, the pole of the line is on the polar of the point.

*Hint.*—Let  $P$  be the given point,  $PC$  the line,  $QS$  the polar of  $P$ .

Draw  $OC$  perpendicular to  $PC$ . Since  $OC.OB = OP.OQ$ ,  $B$  is the pole of  $PC$ .



**53. Cor.**—The line joining two points is the polar of the intersection of their polars; and the point of intersection of two lines is the pole of the line joining their poles.

It follows that the poles of lines which meet in a point are in a straight line, and the polars of points which are in a straight line meet in a point.

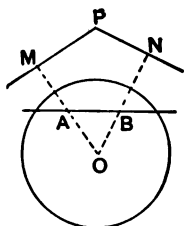


FIG. 53 (1)

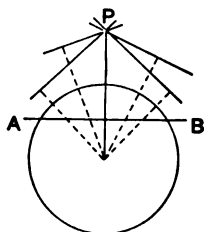


FIG. 53 (2)

Thus if  $PM$  and  $PN$  [Fig. 53 (1)] are the polars of  $A$  and  $B$ ,  $AB$  is the polar of  $P$ , and if  $A$  and  $B$  are the poles of  $PM$  and  $PN$ ,  $P$  is the pole of  $AB$ .

If several lines meet in a point  $P$  [Fig. 53 (2)] their poles are in a straight line  $AB$ , and vice versa.

**54.** The locus of the intersection of tangents to a circle, drawn at the extremities of a chord which passes through a given point, is the polar of the point.

*Hint.*—Let  $P$  be the given point,  $Q$  a point on  $OP$  such that  $OP \cdot OQ = R^2$ , and  $B$  the intersection of the tangents at the extremities of  $TT'$ .

By right triangles  $OC \cdot OB = OT^2$ .

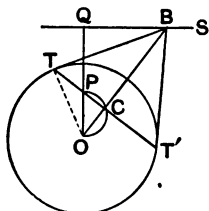


FIG. 54

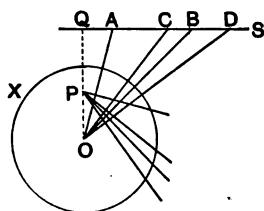


FIG. 55

Hence  $C$  inverts into  $B$  with respect to the centre of inversion  $O$ . But the locus of  $C$  is a circle on  $OP$  as diameter.

Therefore the locus of  $B$  is the straight line  $QS$  perpendicular to  $OQ$ . § 25

**55.** If four points on a straight line form a harmonic system, their four polars form a harmonic pencil.

*Hint.*—Let  $ABCD$  be harmonic points on the line  $QS$ , and  $P$  the pole of  $QS$  with respect to the circle  $X$ .

$OA, OB, OC, OD$  form a harmonic pencil. Also the polars of the four points  $A, B, C, D$  pass through  $P$  and are respectively perpendicular to the rays of this harmonic pencil.

Hence the four polars form a pencil which is equiangular with the pencil  $(O.ABCD)$  and therefore harmonic.

**56.** A line cutting a circle and passing through a fixed point is cut harmonically by the circle, the point, and the polar of the point.

*Hint.*—Let  $P$  be the fixed point,  $LC$  its polar, and  $PM$  the line cutting the circle.

Since  $PO.PC = PA.PB = PM.PN$ , a circle may be circumscribed about the quadrilateral  $OCNM$ .

Hence angle  $OCM = OMN = PCN$ ; then  $CP$  and  $CL$  are the external and internal bisectors of the angle  $MCN$ . Therefore  $P, K, N, M$  are harmonic points. § 334, p. 151

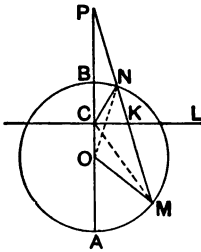


FIG. 56

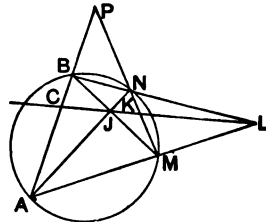


FIG. 57

**57.** A method of drawing the polar of a given point follows from § 56.

*Hint.*—Draw the secants  $PA, PM$ ; also draw  $AM, AN, BM, BN$ .

The line through  $L$  and  $J$  is then the polar of  $P$ .

For  $LMAJNB$  is a complete quadrilateral; and  $A, B, C, P$ , and  $M, N, K, P$ , are therefore two systems of harmonic points. § 14

## NINE POINTS CIRCLE

**58.** The circle through the middle points of the sides of a triangle passes through the feet of the perpendiculars from the opposite vertices, and through the middle points of the segments of the perpendiculars included between their point of intersection and the vertices.

*Hint.*—Let  $ABC$  be the triangle,  $L, M, N$  the middle points of the sides,  $O$  the intersection of perpendiculars,  $X$  the middle point of  $CO$ .

$MX$  is parallel to  $AP$  and consequently perpendicular to  $ML$ . Hence a circle on  $LX$  as diameter passes through  $M$ . For a similar reason it passes through  $N$ .

Since  $LSX$  is a right angle, the circle passes through  $S$ . The circle on  $MY$  as diameter must coincide with this circle since it passes through the points  $L, M, N$ . Hence the circle also passes through  $P$ , etc.

Therefore the circle passes through  $L, M, N, P, R, S, X, Y, Z$ .

**59. Def.**—This circle is the **nine points circle** of the triangle.

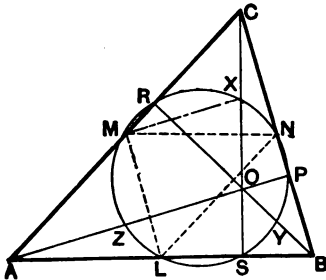


FIG. 58

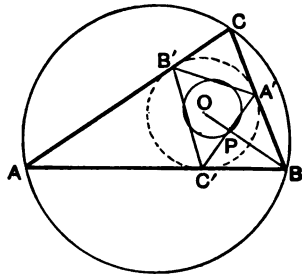


FIG. 60

**60.** The circumscribing circle of a triangle can be inverted into the nine points circle of the triangle formed by joining the points in which the inscribed circle of the original triangle touches the sides. The centre of inversion is the centre of the inscribed circle; the constant of inversion is equal to the square of its radius.

*Hint.*—Since  $OP \cdot OB = OC'^2$ , etc., the vertices  $A, B, C$  invert into the middle points of the sides of the triangle.

Hence the circle through  $A, B, C$  inverts into a circle through the middle points of the sides of the triangle  $A'B'C'$ .

PERSPECTIVE

**61. Def.**—Two figures are in perspective if the lines joining their corresponding points meet in a common point, the centre of perspective.

If the figures are in the same plane, and if, when the lines of the figures are indefinitely produced, the lines joining the corresponding points of intersection meet in a common point, the figures are in plane perspective.

Thus if in Fig. (1) lines  $Aa, Bb, Cc$  meet in a point  $O$ , the triangles  $ABC, abc$  are in perspective.

**62.** If two triangles are in perspective their corresponding sides intersect in points which are in a straight line.

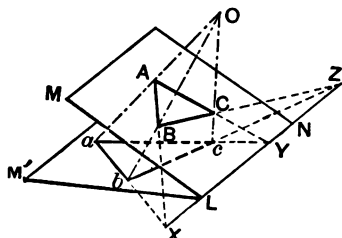


FIG. 62 (1)

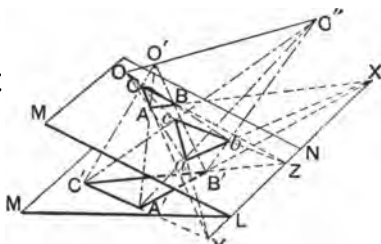


FIG. 62 (2)

(1.) If the triangles are in different planes.

*Hint.*—Let  $O$  be the centre of perspective of  $ABC, abc$ .\*

Since  $AB$  and  $ab$  are both in the plane  $AOB$  they must meet; since  $AB$  is in the plane  $MN$  and  $ab$  in the plane  $M'N'$ , the point of meeting must be in  $LN$  the line of intersection of these planes.

(2.) If the two triangles are in the same plane.

*Hint.*—Draw any line  $OO'O''$  not in the plane of the triangles through the centre of perspective. From any two points  $O', O''$  on this line draw lines through the vertices of the triangles.

$O'A$  and  $O'a$  meet in a point  $A'$  because both are in the plane  $O'OA$ ;

Thus both the triangles  $ABC$  and  $abc$  are projected into  $A'B'C'$ ; hence, their corresponding sides meet on the line of intersection of the plane  $MN$  with the plane of  $A'B'C'$ .

**63. Exercise.**—If two polygons are in perspective their corresponding sides meet in points which are in a straight line.

\* If  $MN$  be a transparent plane and a point of light be at  $O$ , the shadow cast upon the plane  $M'N'$  by the triangle  $ABC$  is the triangle  $abc$ .

**64. Def.**—The line on which the corresponding lines of two figures in perspective meet is the **axis of perspective** of the figures.

**65.** Conversely, if the corresponding sides of two plane triangles intersect in points on a straight line, the triangles are in perspective.

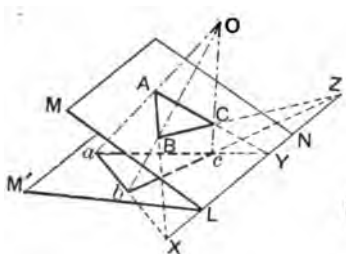


FIG. 65 (1)

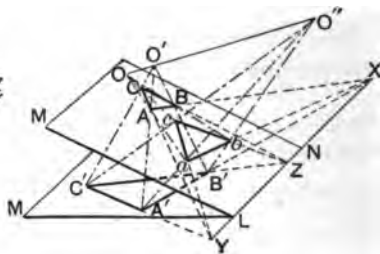


FIG. 65 (2)

(1.) If the triangles are not in the same plane.

*Hint.*—If  $AB$  and  $ab$  meet at  $X$ ,  $Aa$  and  $Bb$  are both in the plane  $AXa$ , and must therefore meet.

Hence  $Aa$ ,  $Bb$ ,  $Cc$  intersect in pairs, and since they are not all three in the same plane, must therefore meet in a point.

(2.) If the triangles are not in the same plane.

*Hint.*—Pass any plane through the line in which the corresponding sides meet and construct in it a triangle in perspective with each of the given triangles [§ 62 (2)]. The line through the centres of perspective,  $O'$ ,  $O''$ , thus found will meet  $Aa$ ,  $Bb$ ,  $Cc$ . Therefore  $Aa$ ,  $Bb$ ,  $Cc$  meet in a point.

**66.** If three triangles are in perspective two by two, and have the same axis of perspective, their three centres of perspective are in a straight line.

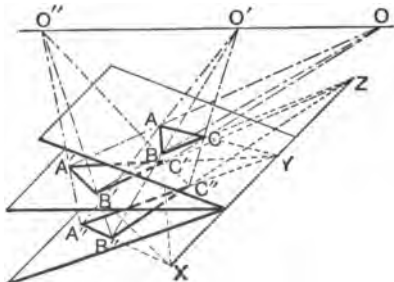


FIG. 66

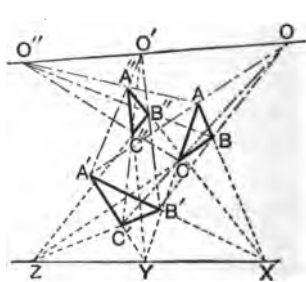


FIG. 66

*Hint.*—Let  $ABC, A'B'C', A''B''C''$  be the triangles, and  $X, Y, Z$  the points in which their corresponding sides meet.

The triangles  $AA'A'', BB'B''$  are in perspective from the centre  $X$ .

Hence the intersections of their corresponding sides are in a straight line. But these intersections are the centres of perspective of the original triangles.

**67. Cor.**—If three triangles are in perspective two by two and have the same axis of perspective, the three triangles formed by joining the corresponding vertices of these triangles are also in perspective two by two and have the same axis of perspective; and the axis of perspective of either system of triangles passes through the centres of perspective of the other system.

**68.** If three triangles are in perspective two by two and have the same centre of perspective, their three axes of perspective meet in a point.

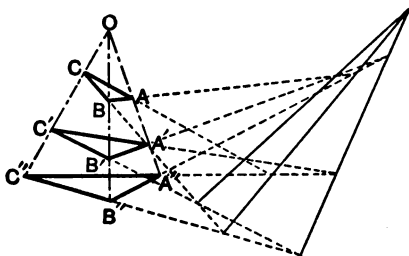


FIG. 68

*Hint.*—Let  $ABC, A'B'C', A''B''C''$  be the triangles and  $O$  their centre of perspective.

The triangles formed by the lines  $AB, A'B', A''B''$  and by the lines  $AC, A'C', A''C''$  are in perspective, since their corresponding sides meet on the line  $AA'$ . Therefore the lines joining their corresponding vertices meet in a point.

**69. Cor.**—If three triangles which are in perspective two by two have the same centre of perspective, the three triangles formed by the corresponding sides of these triangles are also in perspective two by two and have the same centre of perspective; and the three axes of perspective of either system meet in the centre of perspective of the other system.

**70. Exercise.**—Extend the theorems of §§ 66 and 68 to figures other than triangles.

## DUALITY

**71.** If the polar of each point and the pole of each line of a figure be taken, a second figure is formed having a peculiar relation to the first and called its **reciprocal**.

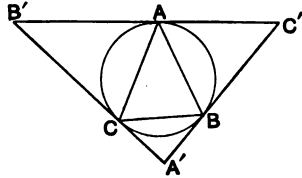


FIG. 71

Thus the triangle  $A'B'C'$  is the reciprocal of  $ABC$ . The sides of  $A'B'C'$  are the polars of the vertices of  $ABC$ ; the vertices of  $A'B'C'$  are the poles of the sides of  $ABC$ .

To a point of the first, corresponds a line of the second.

To a line of the first, corresponds a point of the second.

To points in a straight line in the first, correspond lines through a point in the second. § 53

To lines through a point in the first, correspond points in a straight line in the second. § 53

It follows from these relations, that from a theorem concerning the points and lines of a figure, a reciprocal theorem concerning the lines and points of the reciprocal figure can be inferred.

**72. Def.**—The principle upon which these relations between a figure and its reciprocal depend is called the **principle of duality**.

**73.** The principle of duality in a plane is not necessarily derived from the consideration of poles and polars. A plane figure may be looked upon as composed either of points and the lines joining them, or of lines and their points of intersection, so that the point and line are elements correlative to each other; the relations between reciprocal figures which have already been obtained would follow from this conception.

**74.** Neither is the principle confined to plane figures; in the same way figures in space may be considered as composed either of points or of planes, so that in the geometry of space the point and plane are elements correlative to each other.

It follows, that for reciprocal figures in space :

To a point in the first, corresponds a plane in the second.

To a plane in the first, corresponds a point in the second.

To points in a plane in the first, correspond planes through a point in the second, and vice versa.

To points in a straight line in the first, correspond planes through a straight line in the second, and vice versa.

*Remark.*—In the geometry of space the straight line is correlative to itself.

**75.** Examples of reciprocal theorems of plane geometry.

1. Two points determine a straight line.

2. If the points of intersection of the corresponding sides of two triangles are in a straight line, the lines joining the corresponding vertices of the triangles meet in a point. § 65

3. If three triangles are in perspective two by two and have the same centre of perspective, their three axes of perspective meet in a point. § 68

1. Two straight lines determine a point, their point of intersection.

2. If the lines joining the corresponding vertices of two triangles meet in a point, the corresponding sides of the triangles intersect in points which are in a straight line. § 62

3. If three triangles are in perspective two by two and have the same axis of perspective, their three centres of perspective are in a straight line. § 66

**76.** Examples of reciprocal theorems of the geometry of space.

1. A straight line and a point determine a plane.

2. Three points not in the same straight line determine a plane.

3. Two straight lines which meet in a point are in the same plane.

1. A straight line and a plane determine a point, the point in which the line meets the plane.

2. Three planes which do not pass through the same straight line determine a point.

3. Two straight lines which are in the same plane meet in a point.

#### ANHARMONIC SECTION

**77.** *Def.*—If  $A, B, C, D$  are four points taken in order on a straight line, any one of the following six ratios,



$$\frac{AB \cdot CD}{AD \cdot CB}, \frac{AD \cdot BC}{AC \cdot BD}, \frac{AC \cdot DB}{AB \cdot DC},$$

$$\frac{AD \cdot CB}{AB \cdot CD}, \frac{AC \cdot BD}{AD \cdot BC}, \frac{AB \cdot DC}{AC \cdot DB},$$

is an anharmonic ratio of the points  $A, B, C, D$ .



FIG. 77

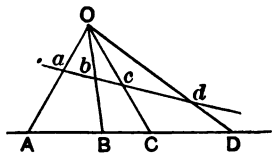


FIG. 78

**78.** If a pencil of four rays cuts two transversals, each anharmonic ratio of the four points of intersection with one transversal is equal to the corresponding ratio of the four points of intersection with the other transversal.

*Hint.*—To prove  $\frac{AB \cdot CD}{AD \cdot CB} = \frac{ab \cdot cd}{ad \cdot cb}$ , etc.

Compare with § 13.

**79. Cor. 1.**—Anharmonic ratios are preserved in perspective.

**80. Def.**—It follows from § 78 that the anharmonic ratios of a pencil of four rays may be defined as the anharmonic ratios of its four points of intersection with a transversal. § 78

**81. Cor. 2.**—If the corresponding rays of two pencils meet on a common transversal, the pencils are equal, that is, have equal anharmonic ratios.

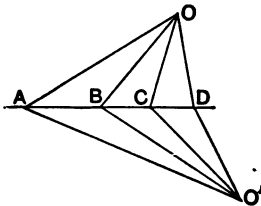


FIG. 81

82. *Cor. 3.*—If two pencils are equal, have a common vertex, and three rays of the first coincide with three rays of the second, the fourth ray of the first coincides with the fourth ray of the second.

83. *Exercise.*—If two pencils have their vertices on a circle and their corresponding rays intersect in points on the circle, the pencils are equal.

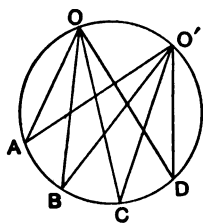


FIG. 83

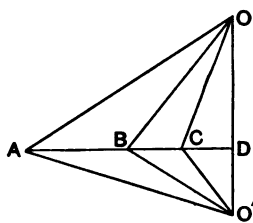


FIG. 84

84. If two equal pencils have a common ray, the intersections of the three remaining pairs of corresponding rays are in a straight line.

*Hint.*—Employ the method of *reductio ad absurdum*.

85. *Exercise.*—Prove by means of § 84 that if two triangles are in plane perspective, the intersections of their corresponding sides are in a straight line.

86. (PASCAL'S THEOREM.) If a hexagon is inscribed in a circle, the intersections of the opposite sides are in a straight line.

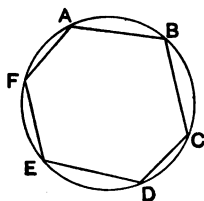


FIG. 86 (r)

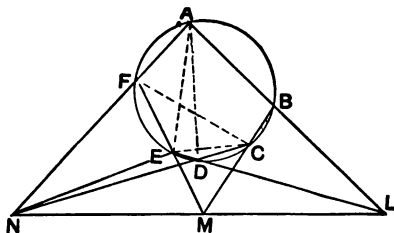


FIG. 86 (s)

*Hint.*—The opposite sides are the 1st and 4th, 2d and 5th, 3d and 6th. Let  $L$ ,  $M$ ,  $N$  be the intersections of the opposite sides.

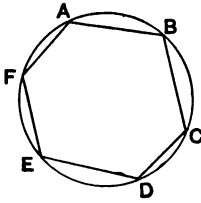


FIG. 86 (1)

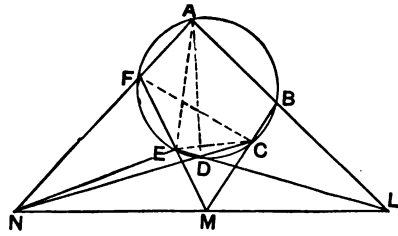


FIG. 86 (2)

Pencil  $\{N.AEDL\} = \{A.NEDL\}$  by § 81,  $= \{C.FEDB\}$  by § 83,  
 $= \{N.AEDM\}$  by § 81.

Therefore  $L, M, N$  are in a straight line. § 82

*Remark.*—This theorem is true of any of the sixty hexagons which can be constructed with six given points as vertices.

**87. Exercise.**—If six points are three by three on two straight lines, the intersections of the opposite sides of a hexagon of which these points are the vertices are in a straight line.

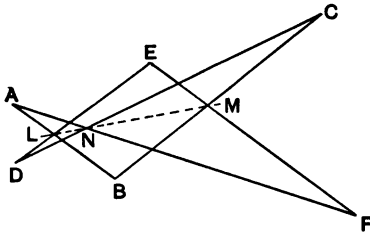


FIG. 87

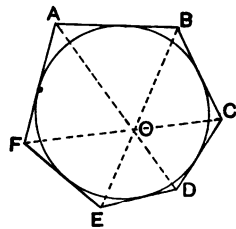


FIG. 88

**88. (BRIANCHON'S THEOREM.)** If a hexagon is circumscribed about a circle, the three lines joining the opposite vertices meet in a point.

*Hint.*—The vertices of the circumscribed hexagon are the poles of the sides of an inscribed hexagon. Therefore this theorem may be inferred from § 86 by the principle of duality.

**89. Exercise.**—If four points are in a straight line, their anharmonic ratio is equal to the anharmonic ratio of their four polars.

*Hint.*—Compare with § 55.

INVOLUTION

**90. Def.**—If the distances of several points,  $A, A',$  etc., in a straight line from a point  $O$  in that line, are connected by the relation

$$OA.OA' = OB.OB' = OC.OC' =$$

the points form a **range in involution**.

**91.** If six points form a range in involution, the anharmonic ratios of any four of the points are equal to the anharmonic ratios of their four conjugates.

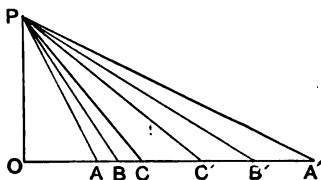


FIG. 91

*Hint.*—At  $O$  erect a perpendicular  $Ol' = \sqrt{OA.OA'}$ . Then  $OP$  is tangent to the circle described through  $A, A', P$ . § 321, p. 145

Hence angle  $OPA = OA'P$ ; likewise angle  $OPB = OB'P$ , etc.

Therefore angle  $APB = A'PB'$ , etc.; that is, the angles of the pencil of four rays  $\{P.AA'BC\}$  are equal to the angles of the pencil  $\{P.A'A'B'C'\}$ .

The anharmonic ratios of the points  $A, A', B, C$  are consequently equal to the anharmonic ratios of the points  $A', A, B', C'$ .

**92. Cor.**—The anharmonic ratios of four points in a straight line are equal to the anharmonic ratios of their inverses, if the centre of inversion is on this line.

**93. Def.**—A pencil of which the rays pass through the points of a range in involution is a **pencil in involution**.

ANTIPARALLELS

**94. Def.**—If two lines are such that the inclination of the first to one side of an angle is equal to the inclination of the second to the other side of the angle, the lines are **antiparallel** to each other with respect to the angle.

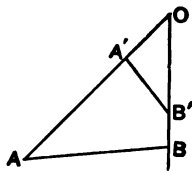


FIG. 94

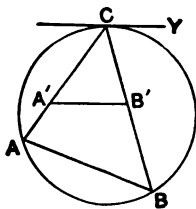


FIG. 95

**95.** An antiparallel to a side of a triangle with respect to the opposite angle is parallel to the tangent to the circumscribing circle drawn at the vertex of that angle.

*Hint.*—Angle  $YCB = CAB = CB'A'$ .

**96. Exercise.**—The lines joining the feet of the perpendiculars of a triangle are antiparallel to the sides with respect to the opposite angles.

## THE GEOMETRICAL AXIOMS

### PLANE, SPHERICAL, AND PSEUDO-SPHERICAL GEOMETRIES

**97.** The geometrical axioms in the Introduction of this Geometry really define the surface on which the theorems of plane geometry are true. This surface is the plane. The axioms also hold true of any surface into which the plane can be bent without stretching, such as the cylinder or cone, provided the definitions of a straight line and parallel lines be modified to apply to these surfaces.

**98.** A sheet of paper may be wrapped about a pencil to form a cylindrical surface; every layer of the paper forms a different part of the surface, and two points that lie in different layers one above the other are separated by the distance which must be traversed to get from one to the other without piercing the paper—that is, by the distance they would be separated in the plane if the paper were unrolled.

**99.** The geometrical axioms are—

(a.) **Straight-line axiom.**—Through every two points there is one and only one straight line.

A straight line of any surface may be defined as the shortest line lying

wholly in the surface which can be drawn between two of its points. Thus, arcs of great circles are the straight lines of the *spherical* surface.

(*b.*) **Parallel axiom.**—Through a given point there is one and only one straight line parallel to a given straight line.

Parallel lines of a surface may be defined as straight lines of that surface which meet at infinity.

(*c.*) **Superposition axiom.**—Any figure in a plane may be freely moved about in the plane without change of size or shape.

This axiom as modified would read :

“Any figure of a surface may be freely moved about in that surface without change of size or shape;” that is, would conform to any portion of the surface without stretching.

**100.** The plane and the surfaces into which it can be bent—the surfaces upon which these axioms hold true—are surfaces of zero curvature.\*

**101.** If the surface or covering of a sphere be detached from the sphere any surface into which it can be bent without stretching is a surface of constant positive curvature. The geometry of such a surface is called **spherical geometry**.

**102.** The superposition axiom is true for the spherical surface.

**103.** The straight-line axiom is true for the spherical surface unless the two points are extremities of a diameter of the sphere, in which case an infinite number of straight lines can be drawn between them.

**104.** There can be no parallel axiom, for on the sphere any two straight lines meet each other at a finite distance.

**105.** In Book VIII. the spherical geometry is developed, not from the axioms which are true on the covering of a sphere independent of the sphere itself, but by considering this covering as belonging to the body in space. This is entirely unnecessary; the spherical surface may be regarded as an independent surface which has no relation to the plane, the straight line, or space. Its geometry may be developed entirely from the axioms which apply to it, just as the geometry of the plane is developed from its axioms.

\* The geometry of such surfaces is called Euclidean Geometry because Euclid first formally stated the axioms as the basis of a geometry.

**106.** All the theorems of "solid geometry" which relate merely to the *surface* of the sphere would be obtained in this way.

**107.** Some of the important differences between spherical geometry and plane geometry are that in spherical geometry—

(a.) All theorems involving parallel lines are lacking.

(b.) The sum of the angles of a triangle is greater than two right angles.

(c.) Figures cannot be similar.

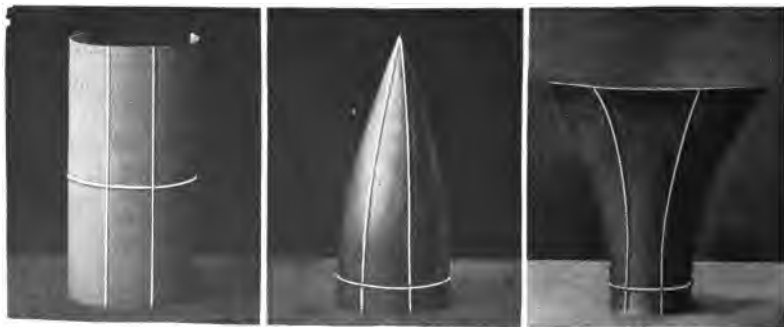
(d.) The area of a polygon is measured by the sum of its angles—that is, by its spherical excess.

**108.** If the surface or covering of a pseudo-sphere be detached from the pseudo-sphere any surface into which it can be bent without stretching is a surface of constant negative curvature. The geometry of such a surface is called **pseudo-spherical geometry**.

**109.** The straight-line axiom and the superposition axiom are true of the pseudo-spherical surface.

**110.** Through a given point of the pseudo-spherical surface two straight lines can be drawn to meet a given straight line at infinity, one meeting it at infinity in each direction. Consequently on this surface the following must be substituted for the parallel axiom:

Through a given point two straight lines can be drawn parallel to a given straight line.



PLANE

SPHERICAL

PSEUDO-SPHERICAL

On this surface two lines perpendicular to the same straight line diverge. The appearance of lines perpendicular to the same line on the plane, spherical, and pseudo-spherical surfaces respectively is shown in the above pictures.

**111.** Pseudo-spherical geometry can be built up from the axioms which are true on the pseudo-spherical surface. Some of the important differences between it and plane geometry are, that in pseudo-spherical geometry—

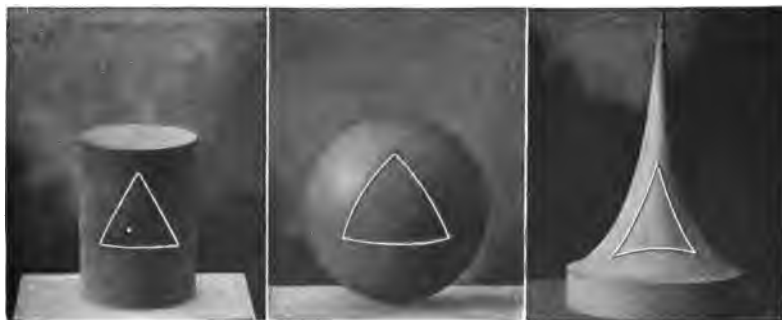
(a.) Theorems which assume that non-parallel lines must meet are not true.

(b.) Theorems involving parallelism must conform to the parallel axiom for a pseudo-spherical surface.

(c.) The sum of the angles of a triangle is less than two right angles.

(d.) Figures cannot be similar.

(e.) The area of a triangle is measured by two right angles less the sum of its angles—that is, by its pseudo-spherical deficiency.



CYLINDER

SPHERE

PSEUDO-SPHERE

**112. Remark.**—The circumference of a circle on the plane surface  $= 2\pi r$ ; on the spherical surface the circumference is less than  $2\pi r$ ; on the pseudo-spherical surface the circumference is greater than  $2\pi r$ . The relation of the areas of circles on the three surfaces is the same.



**113.** There are many theorems which are true in the three kinds of geometry, such as—

The sum of the two adjacent angles which one straight line makes with another straight line is equal to two right angles.

Every point in the perpendicular erected at the middle of a straight line is equally distant from the extremities of that line.

If two angles of a triangle are unequal the sides opposite to them are unequal and the greater side is opposite the greater angle.

The three bisectors of the angles of a triangle meet in a point.

NOTE.—The pseudo-sphere, page 527, is generated by revolving the curve whose equation is

$$y = a \log \frac{a + \sqrt{a^2 - x^2}}{x} - \sqrt{a^2 - x^2}$$

about its  $y$ -axis. The radius of the base of the pseudo-sphere is  $a$ .

TABLE OF MEASURES AND WEIGHTS

*English Measures*

LENGTH

12 inches (in.)	= 1 foot (ft.).
3 feet	= 1 yard (yd.).
5½ yards	= 1 rod (rd.).
4 rods	= 1 chain (ch.).
80 chains	= 1 mile (m.).
1 yard	= .9144 meter.
1 mile	= 1.6093 kilometers.

SURFACE

144 sq. inches	= 1 sq. foot.
9 sq. feet	= 1 sq. yard.
30¼ sq. yards	= 1 sq. rod.
160 sq. rods	= 1 acre.
640 acres	= 1 sq. mile.
1 sq. yard	= 0.8361 sq. meter.
1 acre	= 0.4047 hectare.

VOLUME

1728 cu. inches	= 1 cu. foot.
27 cu. feet	= 1 cu. yard.
128 cu. feet	= 1 cord (cd.).
1 cu. yard	= 0.7646 cu. meter.
1 cord	= 3.625 steras.

ANGLES

60 seconds (")	= 1 minute (').
60 minutes	= 1 degree (°).
90 degrees	= 1 right angle.

CIRCLES

360 degrees	= 1 circumference.
π = 3.1416	= nearly 3¼.

CAPACITY

1 liq. gal.	= 3.785 liters = 23¼ cu. in.
1 dry gal.	= 4.404 liters = 268.8 cu. in.
1 bushel	= 0.3524 hkl. = 2150.42 cu. in.

AVOIRDUPOIS WEIGHT

16 ounces (oz.)	= 1 pound (lb.).
100 lbs.	= 1 hundredweight (cwt.).
20 hundredweight	= 1 ton (T.).

1 pound	= .4536 kilo. = 7000 grains.
1 ton	= .9071 tonneau.

*Metric Measures*

LENGTH

10 millimeters (mm.)	= 1 centimeter (cm.).
10 centimeters	= 1 decimeter (dcm.).
10 decimeters	= 1 meter (m.).
10 meters	= 1 dekameter (dkm.).
10 dekameters	= 1 hektometer (hkm.).
10 hektometers	= 1 kilometer (km.).
1 meter	= 39.37 inches.
1 kilometer	= 0.6214 mile.

SURFACE

100 sq. millimeters	= 1 sq. centimeter.
100 sq. centimeters	= 1 sq. decimeter.
100 sq. decimeters	= { 1 sq. meter. 1 centare (ca.).
100 centares	= 1 are (a.).
100 ares	= 1 nektare (hka.).
1 sq. centimeter	= 0.1550 sq. inch.
1 sq. meter	= 1.196 sq. yards.
1 are	= 3.954 sq. rods.
1 hektare	= 2.471 acres.

VOLUME

1000 cu. millimeters	= 1 cu. centimeter
1000 cu. centimeters	= 1 cu. decimeter.
1000 cu. decimeters	= 1 cu. meter. = 1 stere (st.).
1 cu. centimeter	= 0.061 cu. inch.
1 cu. meter	= 1.308 cu. yards.
1 stere	= 0.2759 cord.

CAPACITY

100 centiliters (cl.)	= 1 liter (l.).
100 liters	= 1 hektoliter (hkl.).
1 liter	= 1.0567 liq. qts. = 1 cu. dcm.

METRIC WEIGHT

1000 grams (gm.)	= 1 kilogram (kilo.).
1000 kilograms	= 1 tonneau (t.).
1 gram	= 15.432 grains.
1 kilogram	= 2.2046 pounds.
1 tonneau	= 1.1023 tons.



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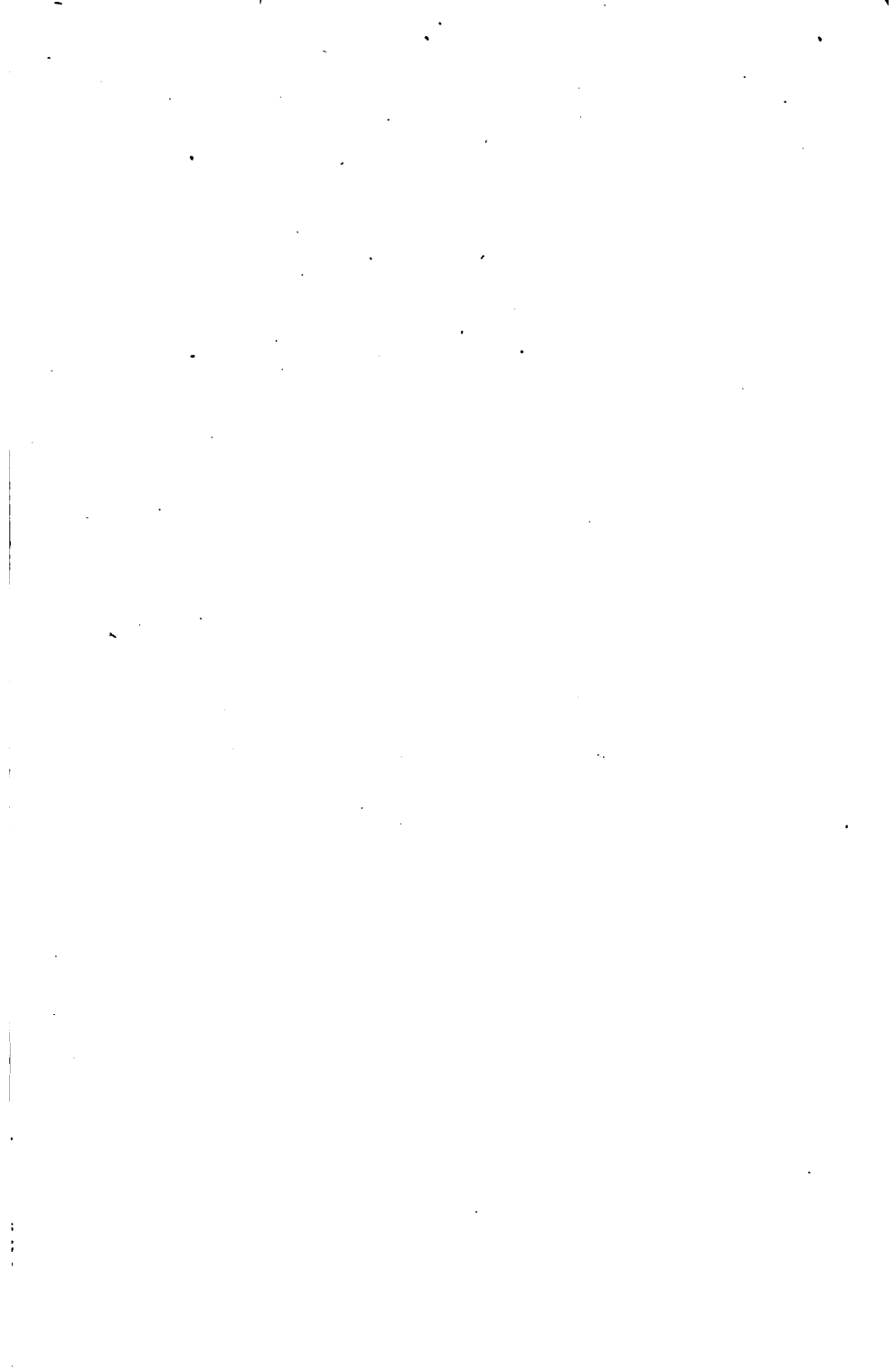
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